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RESEARCH ARTICLE

ESTIMATES OF SECOND HANKEL DETERMINANT FOR SUBCLASSES OF GENERALISED PASCU CLASSES OF FUNCTIONS

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ABSTRACT

A subclass of generalized Pascu classes of functions with respect to symmetric points is considered and obtain sharp upper bounds for the generalized second Hankel determinant $|a_2a_4 - \mu a_3{}^2|$ for an analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ (μ is real and $|z| < 1$) belonging to the class.

INTRODUCTION

Definitions

***Carathéodory Functions** [1]*

Let $\wp$ be the class of analytic functions P(z) of the form

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \tag{1}$$

which satisfies the condition Re $\{P(z)\} > 0$ in the open unit disc $E = \{z: |z| < 1\}$.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

which are analytic in $E = \{z: |z| < 1\}$.

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S is the class of functions of the form (2) which are analytic univalent in E.

***The Hankel Determinant** [6, 7]*

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in E. For $q \geq 1$, the q^{th} Hankel determinant of f is defined by

$$H_q(n) = \begin{vmatrix} a_n a_{n+1} \cdots a_{n+q-1} \\ a_{n+1} a_{n+2} \cdots a_{n+q} \\ \vdots \\ a_{n+q-1} a_{n+q} \cdots a_{n+2q-2} \end{vmatrix}$$

The second Hankel determinant is defined by $|H_2(2)| = \begin{vmatrix} a_2 a_3 \\ a_3 a_4 \end{vmatrix}$. The second Hankel determinant was studied by various authors including Hayman [4] and Pommeranke [8, 9]. We are interested in sharp upper bounds for the functional $|a_2a_4 - \mu a_3^2|$ for certain subclasses of analytic functions.

Sakaguchi [10] introduced the concept of univalent star like functions with respect to symmetric points. A function $f \in A$ is called univalent starlike with respect to symmetric points if and only if

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in E, \tag{3}$$

and the class of functions satisfying (3) may be denoted by S_s^*.

Das and Singh [2] extended the concept of symmetric points to convex and close-to-convex functions. A function $f \in A$ is said to be univalent convex w. r. t. symmetric points if and only if

$$Re\left\{\frac{\left(zf'(z)\right)'}{\left(f(z) - f(-z)\right)'}\right\} > 0, \quad z \in E, \tag{4}$$

and class of such functions is denoted by K_s.
C_s is the class of close-to-convex functions <> f in A with respect to symmetric points if there exists a function

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S_s^* \text{ such that} \tag{5}$$

$$Re\left\{\frac{zf'(z)}{g(z) - g(-z)}\right\} > 0, \quad z \in E. \tag{6}$$

If there exists a function

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in K_s \text{ for which} \tag{7}$$

$$Re\left\{\frac{zf'(z)}{h(z) - h(-z)}\right\} > 0, \quad z \in E, \tag{8}$$

the class of functions <> $f(z)$ in A and satisfying the condition (8) may be denoted by $C_{1(s)}$.
Let C_s^* denote the class of functions in A which satisfy the condition

$$Re\left\{\frac{f(z)}{g(z) - g(-z)}\right\} > 0, \quad g \in S_s^* \text{and <> } z \in E. \tag{9}$$

On replacing g by $h \in K_s$ in (9), the corresponding class of functions f in A may be denoted by $C_{1(s)}^*$.

Let $\alpha \geq 0$ and $\frac{f(z)f'(z)}{z} \neq 0$. Then $C_s^*(\alpha)$ is the class of functions f in A with respect to symmetric points if there exists a function $g \in S_s^*$ such that

$$\left\{\frac{2(1-\alpha)f(z)}{g(z)-g(-z)} + \frac{2\alpha z f'(z)}{g(z)-g(-z)}\right\} = P(z), \quad z \in E. \tag{10}$$

For $h \in K_s$, $C_{1(s)}^*(\alpha)$ is the class of functions f in A which satisfies the condition

$$\left\{\frac{2(1-\alpha)f(z)}{h(z)-h(-z)} + \frac{2\alpha z f'(z)}{h(z)-h(-z)}\right\} = P(z), \quad z \in E. \tag{11}$$

PRELIMINARY LEMMAS

The following lemmas are required to establish our results.

Lemma ([3]). If $P(z) \in \wp$, then $|p_k| \leq 2 (k = 1, 2, 3, \dots)$.

Lemma ([5]). If $P(z) \in \wp$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x, \tag{1}$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z, \tag{2}$$

for some x and z with $|x| \leq 1$ and $|z| \leq 1$.

MAIN RESULTS

Theorem Let $f \in C_s^*(\alpha)$. Then, for any real number μ,

$|a_2 a_4 - \mu a_3^2| \leq$

$$\begin{cases} \frac{8(2B-\mu K)^2}{C(B-\mu K)} - \frac{9\mu}{(1+2\alpha)^2}, & \mu \leq 0; \quad (1) \\ \frac{32}{C}(B-\mu K) + \frac{9\mu}{(1+2\alpha)^2}, & 0 \leq \mu \leq \frac{B}{K}; \quad (2) \\ \frac{9\mu}{(1+2\alpha)^2}, & \frac{B}{K} \leq \mu \leq \frac{2B}{K}; \quad (3) \\ \frac{8(\mu K - 2B)^2}{C(\mu K - B)} + \frac{9\mu}{(1+2\alpha)^2}, & \mu \geq \frac{2B}{K}, \quad (4) \end{cases}$$

Where

$$\begin{cases} B = 4(1+2\alpha)^2 \\ K = 9(1+\alpha)(1+3\alpha) \\ C = 16(1+\alpha)(1+3\alpha)(1+2\alpha)^2 \end{cases} \tag{5}$$

The bounds are sharp.

Proof. Since $f \in C_s^*(\alpha)$, we have

$$(1-\alpha)f(z) + \alpha z f'(z) = G(z)P(z), \quad G(z) = \frac{g(z) - g(-z)}{2}. \tag{6}$$

Equating the coefficients in (6)

$$\begin{cases} a_2 = \dfrac{p_1}{(1+\alpha)} \\ a_3 = \dfrac{p_2 + b_3}{(1+2\alpha)} \\ a_4 = \dfrac{p_3 + p_1 b_3}{(1+3\alpha)} \end{cases} \tag{7}$$

Again g S_s^* implies that
$$zg'(z) = G(z)P(z). \tag{8}$$

Identifying the terms in (8) leads us to

$$\begin{cases} b_2 = \dfrac{p_1}{2} \\ b_3 = \dfrac{p_2}{2} \\ b_4 = \dfrac{p_3}{4} + \dfrac{p_1 p_2}{8} \end{cases} \tag{9}$$

Combination of (7) and (9) give arise to

$$\begin{cases} a_2 = \dfrac{p_1}{(1+\alpha)} \\ a_3 = \dfrac{3p_2}{2(1+2\alpha)} \\ a_4 = \dfrac{2p_3 + p_1 p_2}{2(1+3\alpha)} \end{cases} \tag{10}$$

System (10) yields

$$C(a_2 a_4 - \mu a_3{}^2) = Bp_1(4p_3) + Bp_1{}^2(2p_2) - \mu K(2p_2)^2$$

which on applying lemma 2 can be put in the form

$$C(a_2 a_4 - \mu a_3{}^2) = Bp_1\{p_1{}^3 + 2p_1(4 - p_1{}^2)x - p_1(4 - p_1{}^2)x^2 + 2(4 - p_1{}^2)(1 - |x|^2)z\}$$
$$+Bp_1{}^2\{p_1{}^2 + (4 - p_1{}^2)x\} - \mu K\{p_1{}^2 + (4 - p_1{}^2)x\}^2$$

which implies that

$$C(a_2 a_4 - \mu a_3{}^2) = (2B - \mu K)p_1{}^4 + (3B - 2\mu K)p_1{}^2(4 - p_1{}^2)x \tag{11}$$

$$-(4 - p_1{}^2)\{(B - \mu K)p_1{}^2 + 4\mu K\}x^2 + 2Bp_1(4 - p_1{}^2)(1 - |x|^2)z.$$

Replacing p_1 by p [0, 2] and using triangular inequality, (11) takes the form

$$C|a_2 a_4 - \mu a_3{}^2| = |2B - \mu K|p^4 + |3B - 2\mu K|p^2(4 - p^2)\delta + 2Bp(4 - p^2)(1 - \delta^2) + (4 - p^2)\{|B - \mu K|p^2 + 4|\mu|K\}\delta^2, \quad \delta = |x| \le 1,$$

which can be put in the form

$$C|a_2 a_4 - \mu a_3{}^2| \le \tag{12}$$

$$\begin{cases} (2B-\mu K)p^4+(3B-2\mu K)p^2(4-p^2)\delta+2Bp(4-p^2)+(4-p^2)\{(B-\mu K)p^2-4\mu K-2Bp\}\delta^2 & \text{if } \mu\le 0;\\ (2B-\mu K)p^4+(3B-2\mu K)p^2(4-p^2)\delta+2Bp(4-p^2)+(4-p^2)\{(B-\mu K)p^2+4\mu K-2Bp\}\delta^2 & \text{if } 0\le\mu\le\frac{B}{K};\\ (2B-\mu K)p^4+(3B-2\mu K)p^2(4-p^2)\delta+2Bp(4-p^2)+(4-p^2)\{(\mu K-B)p^2+4\mu K-2Bp\}\delta^2 & \text{if } \frac{B}{K}\le\mu\le\frac{3B}{2K};\\ (2B-\mu K)p^4+(2\mu K-3B)p^2(4-p^2)\delta+2Bp(4-p^2)+(4-p^2)\{(\mu K-B)p^2+4\mu K-2Bp\}\delta^2 & \text{if } \frac{3B}{2K}\le\mu\le\frac{2B}{K};\\ (\mu K-2B)p^4+(2\mu K-3B)p^2(4-p^2)\delta+2Bp(4-p^2)+(4-p^2)\{(\mu K-B)p^2+4\mu K-2Bp\}\delta^2 & \text{if } \mu\ge\frac{2B}{K}. \end{cases} \equiv F(\delta).$$

Since $F'(\delta)\ge 0$, therefore $F(\delta)$ is increasing function in $[0,1]$ and takes its maximum value at $\delta = 1$. Then (12) reduces to

$$C|a_2a_4-\mu a_3^2| \le \tag{13}$$

$$\begin{cases} (2B-\mu K)p^4+(3B-2\mu K)p^2(4-p^2)+(4-p^2)\{(B-\mu K)p^2-4\mu K\} & \text{if } \mu\le 0;\\ (2B-\mu K)p^4+(3B-2\mu K)p^2(4-p^2)+(4-p^2)\{(B-\mu K)p^2+4\mu K\} & \text{if } 0\le\mu\le\frac{B}{K};\\ (2B-\mu K)p^4+(3B-2\mu K)p^2(4-p^2)+(4-p^2)\{(\mu K-B)p^2+4\mu K\} & \text{if } \frac{B}{K}\le\mu\le\frac{3B}{2K};\\ (2B-\mu K)p^4+(2\mu K-3B)p^2(4-p^2)+(4-p^2)\{(\mu K-B)p^2+4\mu K\} & \text{if } \frac{3B}{2K}\le\mu\le\frac{2B}{K};\\ (\mu K-2B)p^4+(2\mu K-3B)p^2(4-p^2)+(4-p^2)\{(\mu K-B)p^2+4\mu K\} & \text{if } \mu\ge\frac{2B}{K}. \end{cases} \equiv G(p). \text{ Then}$$

$$C|a_2a_4-\mu a_3^2| \le \text{Max } G(p). \tag{14}$$

Case (i) $\mu\le 0$.

Then $G(p) = -2(B-\mu K)p^4+8(2B-\mu K)p^2-16\mu K$. On differentiating w.r.t. p, we have

$G'(p) = -8(B-\mu K)p^3+16(2B-\mu K)p$ and $G''(p) = -24(B-\mu K)p^2+16(2B-\mu K)$.

$G'(p)=0$ implies $p=0$ or $\sqrt{\frac{2(2B-\mu K)}{(B-\mu K)}}$. The value $p=0$ gives minimum value of $G(p)$ in which we are not interested. At $p = \sqrt{\frac{2(2B-\mu K)}{(B-\mu K)}}$, $G''(p)<0$ and therefore $G(p)$ is maximum. Moreover max $G(p) = \frac{8(2B-\mu K)^2}{(B-\mu K)}-16\mu K$ which takes us straight to (1).

(1) is sharp for $p_1 = \sqrt{\frac{2(2B-\mu K)}{(B-\mu K)}}$, $p_2=p_1^2-2$ and $p_3=p_1(p_1^2-3)$.

Case (ii) $0\le\mu\le\frac{B}{K}$.

Then $G(p) = -2(B-\mu K)p^4+16(B-\mu K)p^2+16\mu K$. An elementary calculations shows that $\max G(p) = G(2) = 32(B-\mu K)+16\mu K$ which gives (2). The sharp result is obtained on taking $p_1=p_2=p_3=2$.

Case (iii) $\frac{B}{K}\le\mu\le\frac{3B}{2K}$.

Then $G(p) = -8(\mu K-B)p^2+16\mu K$ which is decreasing function of p. Therefore we have max $G(p) = G(0) = 16\mu K$.

Case (iv) $\frac{3B}{2K}\le\mu\le\frac{2B}{K}$.

Then $G(p) = -2(2\mu K-3B)p^4-8(2B-\mu K)p^2+16\mu K$.

In this case also $G(p)\le 16\mu K$.

Combination of cases (iii) and (iv) leads us to (3)

Equality holds in (3) for $p_1=0$, $p_2=-2$ and $p_3=0$.

Case (v) $\mu\ge\frac{2B}{K}$.

Then $G(p) = -2(\mu K - B)p^4 + 8(\mu K - 2B)p^2 + 16\mu K$.

It is easy to show that $G(p)$ is maximum at $p = \sqrt{\frac{2(\mu K-2B)}{(\mu K-B)}}$ and we arrive at (4). The result (4) is sharp for $p_1 = \sqrt{\frac{2(\mu K-2B)}{(\mu K-B)}}$, $p_2 = p_1{}^2 - 2$ and $p_3 = p_1(p_1{}^2 - 3)$. ■

On taking $\alpha = 0$ in the theorem, we obtain

Corollary If $f \in C_s^*$, then

$$|a_2a_4 - \mu a_3{}^2| \le \begin{cases} \dfrac{(8-9\mu)^2}{2(4-9\mu)} - 9\mu, & \mu \le 0; \\ (8-9\mu), & 0 \le \mu \le \dfrac{4}{9}; \\ 9\mu, & \dfrac{4}{9} \le \mu \le \dfrac{8}{9}; \\ \dfrac{(9\mu-8)^2}{2(9\mu-4)} + 9\mu, & \mu \ge \dfrac{8}{9}. \end{cases}$$

Letting $\alpha = 1$ in the theorem, we get

Corollary If $f \in C_s$, then

$$|a_2a_4 - \mu a_3{}^2| \le \begin{cases} \dfrac{(1-\mu)^2}{(1-2\mu)} - \mu, & \mu \le 0; \\ (1-\mu), & 0 \le \mu \le \dfrac{1}{2}; \\ \mu, & \dfrac{1}{2} \le \mu \le 1; \\ \dfrac{(\mu-1)^2}{(2\mu-1)} + \mu, & \mu \ge 1. \end{cases}$$

On the same lines, we can obtain the following

Theorem Let $f \in C_{1(s)}^*(\alpha)$. Then, for any real number μ,

$$|a_2a_4 - \mu a_3{}^2| \le \begin{cases} \dfrac{8(5B-\mu K)^2}{C(3B-\mu K)} - \dfrac{49\mu}{9(1+2\alpha)^2}, & \mu \le 0; \\ \dfrac{8(5B-2\mu K)^2}{C(3B-\mu K)} + \dfrac{49\mu}{9(1+2\alpha)^2}, & 0 \le \mu \le \dfrac{5B}{2K}; \\ \dfrac{49\mu}{9(1+2\alpha)^2}, & \dfrac{5B}{2K} \le \mu \le \dfrac{5B}{K}; \\ \dfrac{8(\mu K-5B)^2}{C(\mu K-3B)} + \dfrac{49\mu}{9(1+2\alpha)^2} & \mu \ge \dfrac{5B}{K}, \end{cases}$$

Where

$$\begin{cases} B = 12(1+2\alpha)^2 \\ K = 49(1+\alpha)(1+3\alpha) \\ C = 144(1+\alpha)(1+3\alpha)(1+2\alpha)^2 \end{cases}$$

On taking $\alpha = 0$ in the theorem, we obtain

Corollary If $f \in C_{1(s)}^*$, then

$$|a_2a_4 - \mu a_3{}^2| \le \begin{cases} \dfrac{(60-49\mu)^2}{18(36-49\mu)} - \dfrac{49\mu}{9}, & \mu \le 0; \\ \dfrac{(60-98\mu)^2}{18(36-49\mu)} + \dfrac{49\mu}{9}, & 0 \le \mu \le \dfrac{30}{49}; \\ \dfrac{49\mu}{9}, & \dfrac{30}{49} \le \mu \le \dfrac{60}{49}; \\ \dfrac{(49\mu-60)^2}{18(49\mu-36)} + \dfrac{49\mu}{9}, & \mu \ge \dfrac{60}{49}. \end{cases}$$

Letting $\alpha = 1$ in the theorem, we get

Corollary If $f \in C_{1(s)}$, then

$$|a_2a_4 - \mu a_3{}^2| \le \begin{cases} \dfrac{(135-98\mu)^2}{324(81-98\mu)} - \dfrac{49\mu}{81}, & \mu \le 0; \\ \dfrac{(135-196\mu)^2}{324(81-98\mu)} + \dfrac{49\mu}{9}, & 0 \le \mu \le \dfrac{135}{196}; \\ \dfrac{49\mu}{81}, & \dfrac{135}{196} \le \mu \le \dfrac{135}{98}; \\ \dfrac{(98\mu-135)^2}{324(98\mu-81)} + \dfrac{49\mu}{81}, & \mu \ge \dfrac{135}{98}. \end{cases}$$

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