ISSN: 0976-3031

Available Online at http://www.recentscientific.com

International Journal of Recent Scientific Research
Vol. 6, Issue, 4, pp.3584-3589, April, 2015

International Journal of Recent Scientific Research

RESEARCH ARTICLE

ANALYSIS OF DISCRETE TIME QUEUES WITH SINGLE SERVER USING CORRELATED TIMES

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ARTICLE INFO

Article History:

Received 14th, March, 2015
Received in revised form 23th, March, 2015
Accepted 13th, April, 2015
Published online 28th, April, 2015

Key words:

Batch service, Correlated times, Discrete time queue, Departure epochs, Packets.

ABSTRACT

An attempt is made in this paper to propose and analyze a distribution of system occupancy in after and before the departure epoch of inter correlated times. The inter arrival times of packets, which arrive one by one follow geometric distribution. The arriving pockets are queued First in First out (FIFO) order. A numerical example illustrates and highlights the findings of the model.

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INTRODUCTION

Discrete-time queueing models have often been used in the performance analysis of ATM multiplexers and switches. In these models, the time axis is assumed to be divided into fixed-length time intervals, referred to as slots, and the servicing of cells is synchronous, i.e. the service of a cell can start or end at slot boundaries only. In the scientific literature, many results can be found with respect to the analysis of discrete-time single-server queues with various types of (uncorrelated or correlated) packet arrival processes and various types of service-time distributions. Packets arrive their inter-arrival time one by one following geometric distribution, wherein the probability of a packet arriving for dispatch in a slot is ‘p’ and its non-arrival is ′q′. If an arriving packet finds the server busy, it joins the queue in first in and first out (FIFO) order. A server transports packets in batches of the minimum number L and maximum K, the service time following common geometric distribution wherein the probability of a batch getting success in a slot is ‘p_1′ and not so is ′q_1′.

The server accesses new arrivals into each ongoing service batch of initial size ‘j’ ($L \leq j < K$), even after service has started on that batch. This is operated till the service time of the ongoing service batch is completed or when the batch being served attains the maximum capacity ‘K’ whichever occurs first. A batch that admits late arrivals as stated above is called the accessible batch. The same concept for accessibility in batches while receiving service was discussed by Gross and Harris (1985). Mathias and Alexander (1998) analyzed the discrete time modeling technique that is mainly applied to the analysis of communication systems. Neuts (1967) presented analytical study of batch service system that operates according to a minimum batch size rule. Kleinrock (1975) discussed for more on continuous time queue with accessible batches. "The inter-arrival and service times are assumed to be geometrically distributed", assumed by Sivasamy (1990). Mathias (1998) derived the inter departure time distribution for batches and for single slot of a discrete time analysis queueing model. It shows that the inter-departure time distribution of a batch service system has characteristic properties that cannot be approximated with sufficient accuracy by other commonly used distribution. Baburaj and Rema (2002) have considered a single server queueing system with a single and batch service. Sivasamy and Pukazhenthi (2009) have analyzed the discrete time bulk service with accessible batch. Baburaj(2010) presented a discrete time bulk service queue under the (a, c, d) policy. Pukazhenthi and Ezhilvanan (2014) discussed the discrete time queue length distribution with a bulk service rule.

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In section 2, the model description and fundamental form of the distribution have been discussed. The steady state distribution of system occupancy at departure epochs obtained in section 3, while the distribution of inter-departure time and number of packets in a batch has been derived in section 4.Brief conclusion is presented in section 5.

Model Description And Fundamental Form Of The Distribution

The queueing model under consideration as pointed out in the preceding section consists of a queue of packets with infinite capacity based on the bulk service rule (L, K). The inter-arrival times A which is the inter- arrival time between the $(n-1)^{th}$ and n^{th} packet are independent and identically distributed random variables with common distribution

$$\{a(k) = \Pr(A_n = k) : k = 0,1,2 \ldots\}, i.e.$$
$$a(k) = pq^{k-1} \; ; \; k = 1,2 \ldots \qquad \ldots (1)$$

Let the probability generating function of A or {a(k)} be denoted as A(z) so that

$$A(z) = \sum_{k=0}^{\infty} a(k) z^k = \frac{pz}{1-qz} \quad \text{for } |z| \leq 1 \qquad \ldots (2)$$

and $A'(z)$ and $A''(z)$ be the first and second derivative of the probability generating function of $A(z)$ and is given by

$$A'(z) = \frac{p}{(1-qz)^2} \qquad and \qquad A''(z) = \frac{2pq}{(1-qz)^3}$$

Hence the mean inter arrival time $E(A)$ and variance $V(A)$ are given by

$$E(A) = A(1) = 1/p$$
$$A''(1) = E(A^2) - E(A) = \frac{2q}{p^2} = \frac{q+1}{p^2} - \frac{1}{p}$$
$$V(A) = \frac{q}{p^2} \qquad \ldots (3)$$

The service times of batches B for the n^{th} batch of packets are independent and identically distributed random variables having common distribution

$$\{b(k) = Pr\,(B_n = k) \; ; \; k = 0,1,2 \ldots\} \text{ so that}$$
$$b(k) = p_1\,(1-p_1)^{k-1} \; ; \; k = 1,2 \ldots \qquad \ldots (4)$$

Let the probability generating function of B or {b(k)} be denoted as B(z) so that

$$B(z) = \sum_{k=0}^{\infty} b(k) z^k = \frac{p_1 z}{1-q_1 z} \quad \text{for } |z| \leq 1 \qquad \ldots (5)$$

As before, the mean service time and variance of service time are given as

$$E(B) = 1/p_1$$
$$B''(1) = E(B^2) - E(B) = \frac{q_1+1}{p_1{}^2} - \frac{1}{p_1}$$
$$V(B) = \frac{q_1}{p_1{}^2} \qquad \ldots (6)$$

Thus load $'\rho'$ of the server is

$$\rho = \frac{E(B)}{KE(A)} = \frac{p}{Kp_1} \qquad \ldots (7)$$

To see the discrete-time queueing system to be stable, the following assumptions are needed for the subsequent analysis.

Each slot is exactly equal to the transmission time of a batch of size K. The time interval $(k, k+1)$ will be referred to as slot $k+1$; $k = 0,1,2, \ldots$. Arriving packets are of fixed size and queued in a buffer of infinite capacity until they enter the server. A packet cannot enter service on its arrival in the slot.

The probability of a packet arriving at the slot is $'p'$ and its non-arrival at that point is $'q'$ which implies that the inter-arrival time is geometrically distribution with parameter $'p'$. A batch of packets of size $j(L \leq j \leq K)$ starts the service at the beginning of a slot, and may only end the service just before the end the of a slot.

The probability $'p_1'$ and the probability of the batch not being transported is $'1 - p_1'$. Service time of a batch is independent of the number of packets in a batch; and the simultaneous occurrence of both arrival and departure in a single slot are ruled out.

All the random variables 'inter-arrival time', 'service time' are independent in the analysis and are non-negative integer valued. The value of the load $'\rho'$ of the server is less than unity *i.e*,

$$\rho = \frac{E(B)}{K\,E(A)} = \frac{p}{Kp_1} < 1$$

Distribution Of System Occupancy At Departure Epochs

Let $X_n \in \{0,1,2 \ldots, \infty\}$ be the number of packets accumulated in the system (queue + service) just after the server has left with n^{th} batch. The steady state distribution $\{x(k) = \lim_{n\to\infty} x_n(k) = \Pr(X_n = k): k = 0,1,2 \ldots \infty\}$ of system occupancy at departure epochs is derived using the embedded Marko Chain (MC) technique.

Let $'G_n'$ be the random variable denoting the number of packets that reach the system during the nth service. Then the distribution $g_n(j)$; $j = 0,1,2, \ldots$ of Gn as n tends to infinity can be served as follows:

$$g(k) = \sum_{m=k+1}^{\infty} \binom{m-1}{k} \{p^k (1-p)^{m-1-k}\} \{p_1 (1-p_1)^{m-1}\} ; \ k = 0,1, \ldots \qquad \ldots (8)$$

From this it is deduced that, for $k = 0,1,2, \ldots \infty$

$$g(k) = \beta(1-\beta)^k \qquad \ldots (9)$$

where

$$\beta = \frac{p_1}{p + p_1 - pp_1} = \frac{p_1}{p_1 + pq_1}$$

More explicitly, we see that,

let $r = m - 1 - k$ in (8) so that $m - 1 = r + k$. Further $m = k + 1$ implies $r = 0$ and $m = \infty$ implies $r = \infty$ thus (8) can be rewritten as,

$$g(k) = \sum_{r=0}^{\infty} \binom{r+k}{k} \{p^k (1-p)^r\} \ \{p_1 (1-p_1)^{r+k}\}$$

$$= \{p^k p_1 (1-p_1) \sum_{r=0}^{\infty} \binom{r+k}{k} \{ (1-p)^r (1-p_1)^r \}$$

$$= \frac{p^k p_1 (1-p_1)^k}{[1 - (1-p)(1-p_1)^{k+1}]} \qquad since \sum_{r=0}^{\infty} \binom{r+k}{k} \ d^r = 1/(1-d)^{k+1}$$

$$= \frac{p^k p_1 (1-p_1)^k}{(p + p_1 - pp_1)^{k+1}}$$

The sequence {Xn} of random variables can be shown to form a Markov Chain (MC) on the state space {0, 1, 2...} with the following one step transition probability matrix $P = (p_{ij})$
where

$$p_{ij} = \begin{cases} \sum_{n=0}^{a-1-i} h_n \sum_{r=0}^{b-a} g_r + \sum_{n=a-i}^{b-i} h_n \sum_{r=0}^{b-n-i} g_r & ; 0 \le i \le a-1 \text{ and } j = 0 \\ \sum_{n=0}^{a-1-i} h_n \, g_{b-a+j} + \sum_{n=a-i}^{b+j-i} h_n \, g_{b-i+j-n} & ; 0 \le i \le a-1 \text{ and } j \ge 0 \\ \sum_{r=0}^{b-i} g_r & ; a \le i \le b \text{ and } j = 0 \\ g_{j+b-i} & ; a \le i \le b \text{ and } j \ge 1 \\ g_{j+b-i} & ; i \ge (b+1) \text{ and } j \ge (i-b) \\ 0 & ; \text{otherwise} \end{cases} \quad \text{... (10)}$$

The unknown probability (row) vector $X_n = (x_0, x_1, x_2, \ldots)$ can be obtained solving the following system of equations:

$$X_n P = X_n , X_n e = 1 \quad \text{... (11)}$$

where $'e'$ denotes the row vector of unities. A number of numerical methods could be suggested to solve the system of equations (11). For example, an algorithm for solving the system (11) is given by Latouche and Ramaswami (1999). Now an upper bound N to 'i' and 'j' of unit step probability function p_{ij} has been selected so that the p_{ij} values are so small for all $i, j \ge N$ which can be ignored. Thus $\sum_{j=0}^{N} p_{ij} = 1$ and $p_{iN} = 1 - \sum_{j=0}^{N-1} p_{ij}$ for all $0 \le i \le N$ and $P = (p_{ij})$ is a square matrix of order$(N+1)$.

Inter-Departure Time and Number of Packets in Batch

Let D_b be the random variable denote the time (number of slots) between the departures of two consecutive batches of packets. Given that atleast L packets are in the system at a departure epoch, the time of the next departure consists only of service time B of the batch being served. If there are not enough packets to make up a batch, i.e. $X < L$, the inter-departure time consists of (i) the inter-arrival time of the (L-X) packets needed to make up a batch and (ii) the service time of that batch. Hence the distribution, $\{d_b(k) = \Pr(D_b = k)\}$ of inter-departure time D_b is

$$D_b(k) = b(k) \sum_{i=l}^{\infty} x(i) + \sum_{j=i}^{L-1} \left\{ \sum_{j=L}^{\infty} h_{j-i}(\,)\,b(k) + \sum_{j=i}^{L-1} h_{j-i} a^{*(L-j)}(\,)\,b(k) \right\} x(i) \quad \text{... (12)}$$

where* stands for convolution operation and a^{*i} denotes i-fold convolution of a(k) with itself. The PGF of D_b is

$$D_b(z) = B(z) \left[\sum_{i-L}^{\infty} x(i) + \sum_{i=0}^{L-1} \left\{ \sum_{j=L}^{\infty} H_{j-i}(z) + \sum_{j=i}^{L-1} H_{j-i}(z) \{A(z)\}^{(L-j)} \right\} x(i) \right]$$

Hence,

$$D'_b(z) = B'(z) \left[\sum_{i-L}^{\infty} x(i) + \sum_{i=0}^{L-1} \left\{ \sum_{j=L}^{\infty} H_{j-i}(z) + \sum_{j=i}^{L-1} H_{j-i}(z) \{A(z)\}^{(L-j)} \right\} x(i) \right]$$

$$+ B(z) \left[\sum_{i=0}^{L-1} \left\{ \sum_{j=L}^{\infty} H'_{j-i}(z) + \sum_{j=i}^{L-1} \{ H'_{j-i}(z) \{A(z)\}^{L-j} + H_{j-i}(z)(L-j)\{A(z)\}^{L-j-1} A'(z) \} \right\} x(i) \right]$$

and the second order derivative,

$$D''_b(z) = B''(z) \left[\sum_{i=k}^{\infty} x(i) + \sum_{i=0}^{L-1} \left\{ \sum_{j=L}^{\infty} H_{j-i}(z) + \sum_{j=i}^{L-1} H_{j-i}(z) \{A(z)\}^{L-j} \right\} x(i) \right]$$

$$+2B'(z) \left[\sum_{i=0}^{L-1} \left\{ \sum_{j=L}^{\infty} H'_{j-i}(z) + \sum_{j=i}^{L-1} \{ H'_{j-i}(z) \{A(z)\}^{L-j} + H_{j-i}(z)(L-j)\{A(z)\}^{L-j-1} A'(z) \} \right\} x(i) \right]$$

$$+ B(z) \left[\sum_{i=0}^{L-1} \left\{ \sum_{j=L}^{\infty} H''_{j-i}(z) \right. \right.$$

$$\left. \left. + \sum_{j=i}^{L-1} \left\{ H''_{j-i}(z) (A(z))^{L-j} + 2H'_{j-i}(z)(L-j)(A(z))^{L-j-1} A'(z) + H_{j-i}(z)(L-j) \left\{ (L-j-1) A^{L-j-2}(z) (A'(z))^2 + A(z)^{L-j-1} A''(z) \right\} \right\} \right\} x(i) \right]$$

Hence,

$$E(D_b) = D'_b(1) = B'^{(1)} + \sum_{i=0}^{L-1}\left\{H'^{(1)} + \sum_{j=i}^{L-1} H_{j-i}(1)(L-j)A'(1)\right\}x(i)$$

$$= E(B) + E(V)\sum_{i=0}^{L-1} x(i) + \sum_{i=0}^{L-i}\left[\left(\frac{p_2}{p_2+pq_2}\right)(L-i)E(A) + \sum_{j=i+1}^{L-1}\left(\frac{p_2}{p_2+pq_2}\right)\left(\frac{pq_2}{p_2+pq_2}\right)(L-i)E(A)\right]$$

where $H'(1) = E(V) = \left(\frac{1}{p_2}\right)$

$$D''_b(1) = B''(1) + 2B'(1)\left[\sum_{i=0}^{L-1}\left\{H'^{(1)} + \sum_{j=i}^{L-1} H_{j-i}(1)(L-j)E(A)\right\}x(i)\right]$$
$$+ B(1)\left[\sum_{i=0}^{L-i}\left[H''^{(1)} + \sum_{J=i}^{L-1}\left\{2H'_{j-i}(1)(L-j)E(A) + H_{j-i}(1)(L-j)\{(L-j-1)A'^{(1)^2} + A''(1)\}\right\}\right]x(i)\right]$$

Let S be a random variable denoting the number of packets in a batch at the beginning of a batch service and $S(k) = Pr(S = s)$ for $k = L, L+1, \ldots K$.

$$S(j) = \begin{cases} y(L), & j = L \\ y(j), & L < j < k \\ \sum_{r=k}^{\infty} y(r), & j = k \end{cases} = \begin{cases} x(L) + \sum_{i=0}^{L-1} x(i)\sum_{r=i}^{L} h(r-j), & j = L \\ x(j) + \sum_{i=0}^{L-1} x(i)h(j-i), & L < j < 1 \\ \sum_{i=k}^{\infty} x(j) + \sum_{i=0}^{L-1} x(i)\sum_{r=k}^{\infty} h(r-i), & j = K \end{cases}$$

The joint distribution $\{f(k,j) = Pr(D_b = k,\ S = j): k \geq 1, L \leq j \leq K\}$ D_b and S is

$$f(k,j) = \partial(k,j)\left[b(k)\sum_{i=k}^{\infty} x(i) + \sum_{i=0}^{L-i} x(i)\sum_{r=k}^{\infty} h_{r-i} * b(k)\right] + \partial(L,j)\left[x(L)b(k) + \sum_{i=0}^{L-i} x(i)\left\{h_{L-i} * b(k) + \sum_{i=0}^{L-i} h_{j-i} * a^{L-j}(\,)b(k)\right\}\right]$$
$$+ [1 - \partial(k-j) - \partial(L,j)]\left[b(k)x(j) + \sum_{i=0}^{L-1}\{h_{j-i}b(k)\}x(i)\right]$$

where $\partial(i,j) = 1$ when $i = j$ and 0 otherwise

The expectation of the random variables D_b ,S and their product D_b ,S given below.

$$E(D_b,S) = E(S)E(B) + K\left\{\sum_{i=0}^{L-i} x(i)\sum_{r=k}^{\infty} H'_{r-i}(1)\right\} + L\sum_{i=0}^{L-i} x(i)\left\{H'_{L-i}(1) + \sum_{j=i}^{L-1}\{H'_{j-i}(1) + h_{j-i}(L-j)E(A)\}\right\} + \left(\sum_{j=L+1}^{K-1} j\right)\left\{\sum_{i=0}^{L-1} H'_{j-i}(1)x(i)\right\}$$

In a similar manner the variance of D_b and S can be calculated .To know how strongly these two random variables depend on each other, the co- efficient of correlation $'r'$ can be computed and it can be defined as,

$$r = \frac{E(D_B S) - E(D_b)E(S)}{\sqrt{Var(D_b)}\sqrt{Var(S)}} \quad \ldots (13)$$

L	E (D_b)	Var (D_b)	E(S)	Var(S)	r
1	5.884	10.999	1.006	.035	-.01716
2	6.923	11.053	2.005	.034	-.02694
3	7.960	11.114	3.005	.034	-.03604
4	8.997	11.185	4.005	.0.34	-.04511
5	10.031	11.268	5.005	.033	-.06323
6	11.063	11.369	6.005	.033	-.07230
7	12.092	11.493	7.005	.032	-.08131
8	13.115	11.649	8.005	.031	-.09026
9	14.132	11.847	9.005	.030	-.09924
10	15.140	12.103	10.005	.029	-.10779
11	16.135	12.436	11.005	.027	-.11619
12	17.114	12.870	12.005	.025	-.12390
13	18.069	13.437	13.005	.023	-.13159
14	18.993	14.173	14.005	.020	-.13789
15	19.874	15.121	15.004	.016	-.14318
16	20.696	16.318	16.004	.013	-.14318
17	21.438	17.784	17.003	.009	-.14715
18	22.070	19.482	18.003	.005	-.15164
19	22.553	21.246	19.001	.002	-.14502
20	22.831	22.632	20.000	.000	.00000

Numerical Study

Consider the values $p = 0.960$, $p_1 = 0.26$ so that $\rho = 0.184615$. For L=1, 2 ...20 below shows the numerical value of the, mean, variance and the co-efficient of correlation r (r) of the stationary queue under study. In the results obtained on $\{x(i),\ i = 0,1, \ldots \infty\}$ sum over 'i' is truncate at 150.

From the above it is observed that $'r'$ value is zero, when L and K is 20 as the batch size constant in this specific case. For L=1 to 19, the values of the co-efficient of correlation $'r'$ are negative. It indicate long inter - departure times correspond to small number in system occupancy at departure epochs to form a batch at the beginning of their service epoch. Mean values of both random variable of D_b,S and variable of D_b increase, with an increase in L. But the variance values of the random variable S decrease, with the value of its increase as the range value between L and K becomes smaller and smaller value.

CONCLUSION

The system occupancy in after and before batches departure epoch of inter correlated time in which the service to customers are provided in batches of the minimum number L and maximum K. The technique used in this paper can be applied to analyze the arrival time of a customer and the busy period of servers. A numerical study has also been carried out, and it has been established that as co-variance between arrival time and service time increase, the queue length also increases.

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