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RESEARCH ARTICLE

ON FINDING INTEGER SOLUTIONS TO HOMOGENEOUS TERNARY QUADRATIC DIOPHANTINE EQUATION

$$x^2+(2k+1)y^2=(k+1)^2z^2$$

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ABSTRACT

Varieties of integer solutions to homogeneous ternary quadratic Diophantine equation represented by $x^2+(2k+1)y^2=(k+1)^2z^2$ are presented.

INTRODUCTION

It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety [1, 2].In particular, the ternary quadratic Diophantine equations in connection with geometrical figures occupy a pivotal role in the orbit of mathematics and have a wealth of historical significance. In this context, one may refer [3-10] for second degree Diophantine equations with three unknowns representing different geometrical figures.

In this paper, the ternary quadratic Diophantine equation representing cone given by $x^2+(2k+1)y^2=(k+1)^2z^2$ is studied for determining its integer solutions successfully through elementary algebra.

Method of analysis

The homogeneous second degree equation in three unknowns to be solved is x^2 + (2k +1) y^2 = (k +1)^2 z^2 (1)

To start with, (1) is satisfied by

$x=4k^3+6k^2+3k, y=2k+1, z=4k^2+2k+1$.

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However, there are many more integer solutions and the process of obtaining various solution patterns is illustrated below :

Process 1

Taking

$$x=(k+1)\,X,\ y=(k+1)\,Y \tag{2}$$

in (1) ,it leads to the ternary quadratic equation

$$X^2+(2k+1)\,Y^2=z^2 \tag{3}$$

which is satisfied by

$$Y=2pq\ ,X=(2k+1)p^2-q^2 \tag{4}$$

and

$$z=(2k+1)\,p^2+q^2 \tag{5}$$

Using (4) in (2) ,we have

$$x=(k+1)\,[(2k+1)\,p^2-q^2\,],\quad y=2(k+1)\,p\,q. \tag{6}$$

Thus , (5) & (6) represent the integer solutions to (1).

Process 2 Consider (3) as the system of double equations as shown below

$$z + X = Y^2$$
$$z - X = 2k + 1$$

Solving the above pair of equations , we have

$$Y = 2s + 1 , X = 2s^2 + 2s - k \quad (7)$$

and

$$z = 2s^2 + 2s + k + 1 \quad (8)$$

From (7) and (2) ,we get

$$x = (k+1)\,(2s^2 + 2s - k)\,, \quad y = (k+1)(2s+1). \quad (9)$$

Thus , (8) & (9) satisfy (1).

Note 1

It is to be noted that ,one may write (3) as the pair of equations as follows:

$$z + X = (2k+1)\,Y^2$$
$$z - X = 1$$

In this case, the solutions to (1) are obtained as

$$x = (k+1)\,[k(2s+1)^2 + 2s^2 + 2s]\,,$$
$$y = (k+1)\,(2s+1)\,,$$
$$z = [k(2s+1)^2 + 2s^2 + 2s + 1].$$

Process 3

The substitution of the transformations

$$x = k(k+1)X, z = (k+1)P + (2k+1)\beta\,, y = (k+1)P + (k+1)^2\beta$$

(10) in (1) leads to the ternary quadratic equation

$$P^2 = X^2 + (2k+1)\beta^2 \quad (11)$$

which is satisfied by

$$\beta = 2pq\,, X = (2k+1)p^2 - q^2\,, P = (2k+1)p^2 + q^2 \quad (12)$$

In view of (10) , the integer solutions to (1) are given b

$$x = k(k+1)[\,2k+1)p^2 - q^2]\,,$$
$$y = (k+1)[\,2k+1)p^2 + q^2] + 2pq\,(k+1)^2\,, \quad (13)$$
$$z = (k+1)[\,2k+1)p^2 + q^2] + 2pq\,(2k+1).$$

Note 2

Apart from (10) ,one may consider the transformations as

$$x = k(k+1)X, z = (k+1)P - (2k+1)\beta\,, y = (k+1)P - (k+1)^2\beta$$

For this choice ,the corresponding integer solutions to (1) are given by

$$x = k(k+1)[\,2k+1)p^2 - q^2]\,,$$
$$y = (k+1)[\,2k+1)p^2 + q^2] - 2pq\,(k+1)^2\,,$$
$$z = (k+1)[\,2k+1)p^2 + q^2] - 2pq\,(2k+1).$$

Process 4

Assume

$$z = a^2 + (2k+1)b^2 \quad (14)$$

Consider

$$(k+1)^2 = (k + i\sqrt{2k+1})\,(k - i\sqrt{2k+1}) \quad (15)$$

Using (14) & (15) in (1) and employing the factorization technique ,we write

$$x + i\sqrt{2k+1}\,y = (k + i\sqrt{2k+1})\,(a + i\sqrt{2k+1}\,b)^2$$

On equating the real and imaginary parts in the above equation ,we have

$$x = k[a^2 - (2k+1)b^2] - 2(2k+1)ab\,,$$
$$y = 2kab + [a^2 - (2k+1)b^2]. \quad (16)$$

Observe that (14) & (16) satisfy (1) .

Process 5

Write (1) as

$$x^2 + (2k+1)\,y^2 = (k+1)^2 z^2 * 1 \quad (17)$$

Express the integer 1 on the R.H.S. of (17) as

$$1 = \frac{(k + i\sqrt{2k+1})\,(k - i\sqrt{2k+1})}{(k+1)^2} \quad (18)$$

Assume

$$z = (k+1)^2[a^2 + (2k+1)b^2] \quad (19)$$

Substituting (15), (18) & (19) in (17) and following the procedure as in Process 4 ,we get

$$x = (k+1)\{(k^2 - 2k - 1)[a^2 - (2k+1)b^2] - 4k(2k+1)ab\}\,,$$
$$y = (k+1)\{2k[a^2 - (2k+1)b^2] + 2(k^2 - 2k - 1)ab\}\,. \quad (20)$$

Thus , (19) & (20) satisfy (1).

Process 6

It is to be observed that , choosing the values of k to be

$$k = 2s^2 + 2s$$

in (1) and employing the transformations

$$x = (2s+1)(2s^2+2s+1)\ X\ , y = (2s^2+2s+1)\ Y , z = (2s+1)w \quad (21)$$

in (1) , it reduces to the Pythagorean equation given by

$$X^2 + Y^2 = w^2 \quad (22)$$

Considering the most cited solutions of (22) and utilizing (21), the corresponding integer solutions to (1) are obtained.

CONCLUSION

In this paper, the homogeneous ternary quadratic equation representing homogeneous cone given by $x^2 + (2k+1)y^2 = (k+1)^2 z^2$ is studied for obtaining its integer solutions through substitution technique and factorization method. One may search for other forms of quadratic equations with multiple variables to determine their integer solutions.

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