INITIAL VALUE PROBLEMS ASSOCIATED WITH FIRST ORDER FUZZY DIFFERENCE SYSTEM - EXISTENCE AND UNIQUENESS

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ABSTRACT

In this paper, we shall be concerned with existence and uniqueness of solutions to first order matrix fuzzy difference system satisfying initial condition of the form \( y(n_0) = y_0 \). The major approach we adopted are the difference inclusions and hence it is unique of its kind.

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INTRODUCTION

Difference equations serve as a natural description of observed evolution phenomena. The growth of biological phenomena are not in general continuous and are in fact discrete in nature. Measurement of data or specified information for an underlying phenomena are in most cases imprecise or partiality specified. That is, each quantity we wish to measure becomes fuzzy valued instead of precise valued. Many of the physical or biological phenomena in nature may not have the exact phenomena or information about their deterministic nature which is prerequisite for the construction of a dynamical system. In general the problem of steering an initial state of the system has become a problem of steering a fuzzy state. In this paper, we shall be concerned with the existence and uniqueness of solutions to first order linear system of difference equation.

Recently Kasi Viswanadh Kanuri and K.N. Murty [3] obtained existence and uniqueness of solutions to first order matrix difference system satisfying boundary conditions at three-points. This paper presents a criteria for the existence and uniqueness criteria for first order fuzzy difference system satisfying initial condition at the initial point.

To establish our main results we make use of the results established on first order difference system by Murty, Spyros, and Viswanadh from [1]. The metrics that we use in this paper are taken from [4]. The results established on initial value problems are taken from [2]. Further results on Fuzzy sets and systems are taken from [5, 6, 7].

Preliminaries

In this section, we introduce notations, definitions and preliminary facts on Fuzzy sets and systems which are used for our later discussion of the paper.

Definition 2.1: Let \( x \) be a non-empty set. A fuzzy set on \( \alpha \in [t_0, t_1] \) is characterized by its membership function \( A: x \rightarrow [t_0, t_1] \) and \( A(n) \) is interpreted as the degree of the membership of element \( n \) in fuzzy set \( A \) for each \( n \in N \) (the set of Natural Numbers).

For each \( 0 \leq \alpha \leq 1 \), we define \( [y]^{\alpha} = \{n \in N : y(n) \geq \alpha \} \). It follows that the \( \alpha \)-level sets \( [y]^{\alpha} \in C(E^n) \).

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It is a well known fact that \( g(y, \bar{y})^\alpha = g[y^\alpha, \bar{y}^\alpha] \) for all \( y, \bar{y} \in E^n \), \( 0 \leq \alpha \leq 1 \) and \( \bar{y} \) is a discrete function.

**Definition 2.2:** Let \( u^{(\alpha)}_i(n) \) be the level set of \( u_i(n) \) then we define
\[
u^{(\alpha)}_i(n) = \{u^{(\alpha)}_i(n), u^{(\alpha)}_i(n), \ldots, u^{(\alpha)}_i(n)\}.
\]

**Definition 2.3:** A fuzzy number in parametric form is represented by \((u^a_0, u^b_0)\), where
\[
u^a_0 \leq \min\{u^a, \nu^b_0\} \text{ and } \nu^b_0 \geq \max\{u^a, \nu^b_0\}, 0 \leq \alpha \leq 1
\]

**Definition 2.4:** We define the zadeh’s extention principle by definition
\[
A[u, v]^\alpha = A[u^\alpha, v^\alpha], 0 \leq \alpha \leq 1
\]

**Definition 2.5:** Let \( u_i(n) \in E^j(i = 1, 2, \ldots, k) \) and define
\[
\hat{u}^{(\alpha)}(n) = (u_1^{(\alpha)}(n), u_2^{(\alpha)}(n), \ldots, u_k^{(\alpha)}(n)) = \hat{\nu}_0^{(\alpha)}(n), \alpha \in (0, 1]
\]

**Main Result**
In this section we shall be concerned with the existence and unicity of solution of the fuzzy first order non-homogenous system.

\[
y_{n+1} = A(n)y_n + b_n \tag{3.1}
\]

where \( A \) is a \((kkk)\) matrix whose components are all continuous functions defined on
\[
N_{+0}^+ \text{and } y_n \in R^k(C_k)
\]

The corresponding homogenous linear difference system is given by

\[
y_{n+1} = A(n)y_n \tag{3.2}
\]

where the initial vector \( y(n_0) = y_0 \) is given Both (3.1) and (3.2) determine the solution uniquely on the set \( N_{+0}^+ \) as can by seen by induction. Let \( r \) be the standard base vector in \( R^k \) and \( y(n, n_0, e_i), i = 1, 2, 3, \ldots, k \) be the linearly independent solutions having \( e_i \) as initial base vector. Let \( S \) be the solution space of (3.2). It may be noted that any element of \( S \) can be expressed as a linear combination of the set of \( k \) solutions of the homogenous system \( y(n, n_0, e_i), i = 1, 2, 3, \ldots, k \).

If \( z_n \) is any solution of the system (3.2) then \( z \) can be expressed as
\[
z_n = \sum_{i=1}^{k} c_iy(n, n_0, e_i)
\]

Consider the following inclusions
\[
y_{n+1} \in A_n\hat{y}_n + b_n
\]

\[
\hat{y}(n_0) \in \hat{y}_0, n_0 \in N.
\]

Let \( \hat{y}(n_0) \) be a solution of (3.2) satisfying \( \hat{y}(n_0) \in \hat{y}_0 \)

Then, we have the following theorem

**Theorem 3.1:**
If \( u \in E^k \), then

1. \( u^{(\alpha)}_i \in p_k[N_{+0}^+ k^k] \) for all \( 0 \leq \alpha \leq 1 \);
2. \( u^{(\alpha)}_i \subset [u]^{(\alpha)}_i \) for all \( 0 \leq \alpha \leq \alpha_2 \leq 1 \);
3. If \( \alpha \) is a non-decreasing sequence converging to \( \alpha > 0 \), then

\[
u^\alpha = \cap_{\alpha \geq 1} \nu^{(\alpha)}_i
\]

Conversely, if \( \nu^\alpha : 0 \leq \alpha \leq 1 \) is a family of subsets of \( R^k \) satisfying (1) - (3) above, then there exists a \( u \in E^k \) such that
\[
u^\alpha = A^\alpha \text{ for } 0 \leq \alpha \leq 1 \text{ and } \nu^0 = u_0^\alpha \in A^\alpha \text{ for } 0 \leq \alpha \leq 1
\]

Proof: For the proof of the theorem, we refer Negoita and Ralescu [5].

We now consider the following inclusions
\[
y_{n+1} \in A(n)y_n + b_n \tag{3.3}
\]

\[
y(n_0) \in y_0 \tag{3.4}
\]

Let \( \hat{y}_n \) be the solution of (3.3) satisfying (3.4). We have the following lemma.

**Lemma 3.1:**
Let \( \hat{y}(n) \in p_k[N_{+0}^+ N_0] \), for every \( 0 \leq \alpha \leq 1 \), \( n \in [0, N] \)

Proof: First we observe that \( \hat{y}_n \) is a nonempty set since \( \nu^\alpha(n) \) has a measurable selection. By choosing \( k = \max_{\nu \in [0, N]} ||\phi(n)|| \), where \( \phi \) is a fundamental solution of

\[
y_{n+1} = A(n)y_n, L = \max_{\nu \in [0, N]} ||I_k|| = 1
\]

\[
M_1 = \max_{\nu \in [0, N]} ||u_0||, \nu_0 \in \nu(n), n \in [0, N], \text{ and } M_2 = \max_{\nu \in [0, N]} ||b_0|| \text{ and if for any } \hat{y} \in \nu^\alpha, \nu \in [0, N]
\]

\[
y(n) = \hat{y}_n + \sum_{j=0}^{n-1} \nu^\alpha(n - j - 1)b_nu(j)
\]

By taking suitable norm as in [4], we get

\[
||\hat{y}(n)|| \leq KL||\nu|| + KLM_1M_2
\]

Hence \( ||\hat{y}(n)|| \) is bounded. For any \( n_1, n_2 \in [0, N] \), consider

\[
||\hat{y}(n_1) - \hat{y}(n_2)|| \leq ||\phi(n_1) - \phi(n_2)||\hat{y}_0 + KLM_1M_2||n_1 - n_2|| + M_1M_2\sum_{j=0}^{k-1} \nu^\alpha(n - j - 1)b_nu(k)
\]

Since \( u_k \in \nu^\alpha(n) \) is closed, there exists a subsequence of \( \sum u_k \) converging weakly to \( u \in \nu^\alpha(n) \).

From Mazur’s theorem [5], there exists a sequence of numbers \( \lambda_i > 0, \sum \lambda_i = 1 \) such that \( \sum \lambda_i u_k \) converges strongly to \( u \). Thus from (3.7), we have

\[
\sum \lambda_i \hat{y}_k(n) = \sum \lambda_i \phi(n)\hat{y}_0 + \sum_{j=0}^{k-1} \phi(n - j - 1)b_nu_k(j)
\]

As \( n \to \infty, (3.6) \) and Fatou’s lemma it follows that

\[
\hat{y}(n) = \phi(n)\hat{y}(0) + b_nu_j.
\]
Thus \( \breve{y}(n) \in \breve{y}^a \) and \( \breve{y}^a \) is closed. Let \( \hat{y}_1 \) and \( \hat{y}_2 \) \( \in \breve{y}_a \). Then there exists \( u_1 \) and \( u_2 \ \in \breve{u}_a(n) \), such that
\[
\hat{y}_1(n + 1) = A(n)\hat{y}_1(n) + b_nu_1(n)
\]
and
\[
\hat{y}_2(n + 1) = A(n)\hat{y}_2(n) + b_nu_2(n).
\]
Let \( \breve{y}(n) = \lambda\hat{y}_1(n) + (1 - \lambda)\hat{y}_2(n) \), \( 0 \leq \lambda \leq 1 \) then \( \breve{y}(n + 1) = A(n)[\lambda\hat{y}_1(n) + (1 - \lambda)\hat{y}_2(n)] + b_n[Au_1(n) + (1 - \lambda)u_2(n)] \)

Since \( u^a(n) \) is convex, \( \lambda_1 u_1(n) + (1 - \lambda)_u_2(n) \in \breve{u}^a(n) \)

We have \( \breve{y}(n + 1) = A(n)\breve{y}(n) + b_nu^a(n) \). Hence \( \breve{y} \in \breve{y}^a \). Thus \( \breve{y}^a \) is convex. Therefore \( \breve{y}^a \) is non-empty, compact and convex in \( [C[0,N], \{N^+\}]^k \).

Thus from Ascoli’s Lemma it follows that \( [\breve{y}(n)]^a \) is convex in \( \{N^+\}^k \) for every \( n \in [0,N] \). Therefore the result follows.

**Lemma 3.2:** If \( \alpha_k \) be a non-decreasing sequence converging to \( \alpha > 0 \), then
\[
\breve{y}^a(n) = \bigcap_{k \geq 1} \breve{y}^{a_k}(n)
\]

**Proof:**
Let \( \breve{u}^{a_k}(n) = \breve{u}_1^{a_k}x_1^a x_1^a x \cdots x_k^a \)
and
\[
\breve{u}_k^a(n) = \breve{u}_1^ax_1^a x_1^a x \cdots x_k^a
\]
Consider the inclusions
\[
\breve{y}(n + 1) \in A(n)\breve{y}(n) + b_n\breve{u}^{a_k}(n) \quad (3.7)
\]
\[
\breve{y}(n + 1) \in A_n\breve{y}(n) + b_n\breve{u}^a(n) \quad (3.8)
\]
Let \( \breve{u}^{a_k} \) and \( \breve{u}^a \) be the solution sets of (3.7) and (3.8) respectively. Since \( \mu_i(n) \) is a fuzzy set and from Theorem 3.1, we have
\[
u_i^a = \bigcap_{k \geq 1} \breve{u}_i^{a_k}
\]
consider
\[
\breve{u}^a(n) = \breve{u}_1^ax_1^ax_1^a \cdots x_k^a
\]
\[
= \bigcap_{k \geq 1} \breve{u}_1^{a_k}x_1^{a_k} X \cdots x_k^{a_k}
\]
\[
\geq \bigcap_{k \geq 1} \breve{u}_k^{a_k}(n)
\]
and then
\[
\breve{u}_1(n + 1) \in A(n)\breve{u}(n) + b_n\breve{u}^a(n)
\]
\[
A(n)\breve{y}_1(n) + b_n\breve{u}^{a_k}(n), k = 1,2,\cdots
\]
Thus, we have \( \breve{u}^a \subset \breve{u}^{a_k}, k = 1,2,\cdots \) which implies
\[
\breve{y}^a \subset \bigcap_{k \geq 1} \breve{y}^{a_k} \quad (3.9)
\]

Let \( \breve{y} \) be the solution set of the inclusion (3.7) for \( k \geq 1 \)

Then \( \breve{y}(n) \in \phi_n\breve{y}_0 + b_n\breve{u}^{a_k}(n) \).

**References**


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