INTRODUCTION

A partition of a positive integer \( n \) is a finite non-increasing sequence of positive integers \( \lambda_1, \lambda_2, \ldots, \lambda_r \) such that
\[
\sum_{i=1}^{r} \lambda_i = n
\]
and is denoted by \( n = (\lambda_1, \lambda_2, \ldots, \lambda_r) \).

\( n = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_r \) or \( \lambda = (\lambda^{f_1}_1, \lambda^{f_2}_2, \lambda^{f_3}_3, \ldots) \) when \( \lambda_i \) repeats \( f_i \) times, \( \lambda_2 \) repeats \( f_2 \) times and so on. The \( \lambda_i \) are called the parts of the partition. In what follows \( \lambda \) stands for a partition of \( n \), \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \), \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \). The set of all partitions of \( n \) is represented by \( \xi(n) \) by and its cardinality \( p(n) \).

If \( 1 \leq r \leq n \) then \( \xi_r(n) \) is the set of partitions of \( n \) with \( r \) parts and its cardinality is denoted by \( p_r(n) \). A partition of \( n \) with exactly \( r \) parts is called \( r \)-partition of \( n \). We define
\[
p_r(n) = \begin{cases} 
0 & \text{if } r = 0 \text{ or } r > n \\
\text{number of } r-\text{partitions of } n & \text{if } 0 < r \leq n
\end{cases}
\]

\( spt(n) \) denotes the number of smallest parts including repetitions in all partitions of \( n \). \( spt_r(n) \) denotes the number of \( r \)-smallest parts in all partitions of \( n \).

The number of \( r - \text{partitions of } n \) with least part greater than or equal to \( k \) is represented by \( p(k, n) \).

**Existing generating functions are given below.**

<table>
<thead>
<tr>
<th>Function</th>
<th>Generating function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_r(n) )</td>
<td>( \frac{q^r}{(q)_r} )</td>
</tr>
<tr>
<td>( p_r(n-k) )</td>
<td>( \frac{q^{r+k}}{(q)_r} )</td>
</tr>
<tr>
<td>number of divisors</td>
<td>( \sum_{q=0}^{\infty} \frac{q^a}{(1-q^a)} )</td>
</tr>
<tr>
<td>sum of divisors</td>
<td></td>
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Theorem: The generating function for the sum of smallest parts of the partitions of $n$ is

\[
\sum_{n=1}^{\infty} \text{sum spt}_n(q)^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \text{sum spt}_n(q)_{n-1} (1-q^n)
\]

Proof: From [4] we have the sum of smallest parts $\text{sum spt}_n$ of the partitions of a positive integer $n$ is

\[
\text{sum spt}_n = \sum_{k=1}^{\infty} k (p(k, n-k, t) + \sigma(n))
\]

where $\sigma(n)$ is the sum of positive divisors of $n$.

First replace $k+1$ by $k$, then replace $n$ by $n-k$ in [3]

\[
= \sum_{k=1}^{\infty} \sum_{r=1}^{k} \sum_{k'} p_r (n, n-k, r-k-1) + \sigma(n)
\]

Hence the generating function for the sum of smallest parts of the partitions of a positive integer $n$ is

\[
\sum_{n=1}^{\infty} \text{sum spt}_n(q)^n = \sum_{k=1}^{\infty} \sum_{r=1}^{k} \sum_{r=1}^{\infty} k q^{r+k+r(k-1)} (q)_r (1-q^k)
\]

from (1.1.1)

\[
= \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} k q^{r+k} (q)_r (1-q^k)
\]

\[
= \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} k q^{r+k} \left[ \sum_{r=1}^{\infty} \left(\frac{q^k}{r} \right) \right] + \sum_{k=1}^{\infty} k q^k (1-q^k)
\]

\[
= \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} k q^{r+k} \left[ 1 + \sum_{r=1}^{\infty} \left(\frac{q^k}{r} \right) - 1 \right] + \sum_{k=1}^{\infty} k q^k (1-q^k)
\]

\[
= \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} k q^{r+k} \prod_{r=0}^{\infty} \left( \frac{1}{1-q^k q^r} \right)
\]

Theorem: The sum $r - \text{spt}_i(n)$ of $i^{th}$ smallest parts of $r - \text{partitions of } n$ is

\[
\sum_{n=1}^{\infty} (r - \text{spt}_i(n)) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \text{sum spt}_n(q)_{n-1} (1-q^n)
\]

Proof: Let $n = (\lambda_1, \lambda_2, \ldots, \lambda_r) = (\mu^a)$ be any $r$ - partition of $n$ with all parts equal.

We know that the sum of the smallest parts of $r - \text{partitions of } n$ such that the smallest part is the first part (i.e. $\lambda_1$ as smallest part) and having $k$ as smallest part is $k^\beta$

where $\beta = \begin{cases} 1 & \text{if } \frac{n}{r} = k \\ 0 & \text{otherwise} \end{cases}$

The generating function for the sum of smallest parts of $r - \text{partitions of } n$ such that smallest part as the first part (i.e. $\lambda_1$ as smallest part)
\[ \sum_{n=1}^{\infty} (r - \text{spt}_1(n)) q^n = \sum_{k=1}^{\infty} k q^{r^2} \left(1 - q^r\right)^2 \] for \( r = 1 \)

Let \( n = (\lambda_1, \lambda_2, \ldots, \lambda_r) = (\mu_1, \mu_2, \ldots, \mu_r) \) be any \( r - \text{partition of } n \) with two distinct parts.

Subtracting \( \mu_k \) from each \( \lambda_i \) for \( i = 1 \) to \( r \), we get

\[ n_i = (\mu_i^{(0)}) \] where \( n_i = n - r \mu_k, \ r_i = r - \alpha_k \), and \( \mu_i^{(0)} = \mu_i - \mu_k \)

The sum of the smallest parts of \( n_i - \text{partitions of } n \) such that smallest part is the first part and having \( k \) as the smallest part is \( k \beta_1 \).

where \( \beta_1 = \begin{cases} 1 & \text{if } r \mid n_i, \text{ and } n = n_1 + r \mu_2, \\ 0 & \text{otherwise} \end{cases} \)

Then the sum of second smallest parts of \( r - \text{partitions of } n \) such that second smallest part is the first part and having \( k \) as the smallest part is \( k \beta_1 \).

where \( \beta_1 = \begin{cases} 1 & \text{if } \frac{n-r \mu_2}{r_1} = n_1 + r \mu_2, \\ 0 & \text{otherwise} \end{cases} \)

Therefore the generating function for the sum of the second smallest parts of \( r - \text{partitions of } n \) such that the second smallest part as the first part (i.e \( \lambda_1 \) as second smallest part) is

\[ \sum_{n=1}^{\infty} \sum_{r=1}^{n-1} \sum_{\mu_1}^{\mu_2} k q^{\mu_1 r + (k-\mu_2)} \]

\[ = \sum_{r=1}^{\infty} \sum_{\mu_1}^{\mu_2} k q^{\mu_1 r + (k-\mu_2)} \sum_{r=1}^{n} \sum_{\mu_1}^{\mu_2} \mu_1 q^{\mu_1 r} \]

\[ = \left[ \sum_{r=1}^{\infty} \mu_2 q^{\mu_1 r} \sum_{r=1}^{n} \sum_{\mu_1}^{\mu_2} q^{\mu_1} + \sum_{r=1}^{\infty} q^{\mu_1 r} \sum_{r=1}^{n} \sum_{\mu_1}^{\mu_2} \mu_1 q^{\mu_1 r} \right] \]

\[ = \left[ \frac{q^r}{(1-q^r)^2} \sum_{r=1}^{n} q^r \sum_{r=1}^{\infty} \left( \frac{1}{1-q^r} \right)^2 \right] \]

for \( r = 2 \) (1.3.4)

Continuing this process, we get the generating function for the sum of \( i^{th} \) smallest parts of \( r - \text{partitions of } n \) such that \( i^{th} \) smallest part as first part (i.e \( \lambda_i \) as \( i^{th} \) smallest part) is

\[ \sum_{n=1}^{\infty} \sum_{r=1}^{n} \sum_{\mu_1}^{\mu_2} k q^{\mu_1 (r-i) + (k-\mu_2) + i} \]

\[ = \sum_{k=1}^{\infty} k q^{\mu_1} \frac{q^r}{(1-q^r)^2} \sum_{r=1}^{n} \sum_{\mu_1}^{\mu_2} \left( \frac{1}{1-q^r} \right)^2 \]

\[ = \left[ \sum_{k=1}^{\infty} \sum_{r=1}^{n} \frac{q^r}{(1-q^r)^2} \right] \]

\[ = \sum_{i=0}^{\infty} \left( \frac{q^r}{(1-q^r)^2} \right) \]

\[ = \frac{q^r}{(1-q^r)^2} \sum_{i=1}^{\infty} \left( \frac{1}{1-q^r} \right)^2 \]

(1.4) Theorem: The generating function for the sum of \( i^{th} \) smallest parts of \( r - \text{partitions of } n \) is

\[ \sum_{n=1}^{\infty} (r - \text{spt}_i(n)) q^n = \left\{ \begin{array}{ll}
q^r & \frac{q^r}{(1-q^r)^2} \sum_{r=1}^{n} \sum_{\mu_1}^{\mu_2} k q^{\mu_1 (r-i) + (k-\mu_2) + i} \left( \frac{1}{1-q^r} \right)^2 \\
+ q^r \sum_{i=r}^{n} \sum_{\mu_1}^{\mu_2} k q^{\mu_1 (r-i) + (k-\mu_2) + i} \left( \frac{1}{1-q^r} \right)^2 & \text{for } i = r \quad \blacksquare
\end{array} \right. \]

Proof: The sum of smallest parts of \( r - \text{partitions of } n \) having \( k \) as a smallest part is

\[ k \sum_{i=0}^{\infty} p_{r-1-i} \left( n - (k-1)r - 1 - i \right) + k \beta \]

where \( \beta = \begin{cases} 1 & \text{if } \frac{n}{r} = k \\ 0 & \text{otherwise} \end{cases} \)

The generating function for the sum of the smallest parts of \( r - \text{partitions of } n \) is

\[ \sum_{n=1}^{\infty} \sum_{r=1}^{n} \frac{q^r}{(1-q^r)^2} \]

\[ = \sum_{k=1}^{\infty} k q^{\mu_1} \frac{q^r}{(1-q^r)^2} \sum_{r=1}^{n} \sum_{\mu_1}^{\mu_2} \left( \frac{1}{1-q^r} \right)^2 \]

\[ = \sum_{i=0}^{\infty} \left( \frac{q^r}{(1-q^r)^2} \right) \sum_{i=1}^{\infty} \left( \frac{1}{1-q^r} \right)^2 \]

\[ = \frac{q^r}{(1-q^r)^2} \sum_{i=1}^{\infty} \left( \frac{1}{1-q^r} \right)^2 \]

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\[
\begin{align*}
&= \frac{q^r}{(1-q^r)} \left( \sum_{i=0}^{\infty} \frac{1}{(q)_i} \right) + 1 \\
&= \frac{q^r}{(1-q^r)} \left( \sum_{n=1}^{\infty} \frac{1}{(q)_n} \right) + 1
\end{align*}
\]

From theorems (3.3.1) and (3.3.4) in [5], the sum of second smallest parts (without $\lambda_i$) of $r-\text{partitions of } n$ with least part $k$ is

\[
\sum_{i=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k \cdot p_{n-1-1} \left( n - \mu_r - (k - \mu_i - 1) r_1 - 1 - i \right)
\]

The generating function for the sum of second smallest parts (without $\lambda_i$) of $r-\text{partitions of } n$ is

\[
\sum_{\mu_i=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k \left( q^{\mu_i(r-1)+k-\mu_i n} \right) \left( (q)_{n-1-i} \right)
\]

\[
= \sum_{\mu_i=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k \left( q^{\mu_i(r-1)+k-\mu_i n} \right) \left( (q)_{n-1-i} \right)
\]

Similarly, the generating function for the sum of third smallest parts of $r-\text{partitions of } n$ is

\[
\sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{\mu_i=1}^{\infty} \left( \mu_i + \mu_i - 1 \right) q^{\mu_i(r-1)+\mu_i - 1} \left( (q)_{n-1-i} \right)
\]

where $k = \mu_i + \mu_i - 1$

By induction, we get the generating function for the sum of the $i^{th}$ smallest parts of $r-\text{partitions of } n$ which is given by

\[
\sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{\mu_i=1}^{\infty} \left( \mu_i + \mu_i - 1 \right) q^{\mu_i(r-1)+\mu_i - 1} \left( (q)_{n-1-i} \right)
\]

The generating function for the sum of smallest parts of $r-\text{partitions of } n$ are equal to

\[
\frac{q^r}{(1-q^r)} \sum_{n=1}^{\infty} \frac{1}{(q)_n} + \frac{q^r}{(1-q^r)} \sum_{n=1}^{\infty} \frac{1}{(q)_n}
\]

for $r = 2$

Therefore the generating function for the sum of second smallest parts of $r-\text{partitions of } n$ is

\[
\sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{\mu_i=1}^{\infty} \left( \mu_i + \mu_i - 1 \right) q^{\mu_i(r-1)+\mu_i - 1} \left( (q)_{n-1-i} \right)
\]

\[
= \frac{q^r}{(1-q^r)} \sum_{n=1}^{\infty} \frac{1}{(q)_n} + \frac{q^r}{(1-q^r)} \sum_{n=1}^{\infty} \frac{1}{(q)_n}
\]

Similarly the generating function for the sum of third smallest parts of $r-\text{partitions of } n$ is

\[
\sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{\mu_i=1}^{\infty} \left( \mu_i + \mu_i - 1 \right) q^{\mu_i(r-1)+\mu_i - 1} \left( (q)_{n-1-i} \right)
\]

\[
= \frac{q^r}{(1-q^r)} \sum_{n=1}^{\infty} \frac{1}{(q)_n} + \frac{q^r}{(1-q^r)} \sum_{n=1}^{\infty} \frac{1}{(q)_n}
\]

References