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*International Journal of Recent Scientific Research*
*Vol. 4, Issue, 5, pp. 576 - 578, May, 2013*

**International Journal of Recent Scientific Research**

ISSN: 0976-3031

**RESEARCH ARTICLE**

# SECONDARY k-NORMAL MATRICES

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## ARTICLE INFO

***Article History:***

Received 15th, March, 2013
Received in revised form 17th, April, 2013
Accepted 24th, May, 2013
Published online 28th May, 2013

***Key words:***

Normal, s-k normal and Moore-Penrose inverse,
AMs Classification: 15A09, 15A57

## ABSTRACT

The concept of s-k normal matrices is introduced. Characterizations of s-k normal matrices are obtained.



## INTRODUCTION

A square complex matrix $A \in C_{nxn}$ is called normal if $AA^* = A^*A$, where $A^* = \overline{A}^T$ denotes the conjugate transpose of A[2]. There are many equivalent conditions in the literature for a square matrix to be normal [3]. For an mxn complex matrix A, the Moore Penrose inverse $A^\dagger$ of A[2] is the unique nxm matrix X satisfying the following four Penrose equations:

(i) $AXA = A$ (ii) $XAX = X$

(iii) $(AX)^* = AX$ (iv) $(XA)^* = XA$. [2] Recently, Hill and Waters [3] have developed a theory for k-real and k-hermitian matrices. Ann Lee [1] has initiated the study of Secondary symmetric matrices, that is matrices whose entries are symmetric about the (Skew) Secondary diagonal. Ann Lee has shown that for complex matrix A, the usual transpose $A^T$ and Secondary transpose $A^S$ are related as $A^S = VA^TV$ where 'V' is the permutation matrix with units in the secondary diagonal. The concept of s-normal matrices is introduced by S.Krishnamoorthy & R.Vijayakumar [6] and the concept of k-normal matrices introduced by S.Krishnamoorthy and R.Subhash [7]. In this paper characterization of s-k normal matrices are discussed.

### *2. Preliminaries and Notations*

Let Cnxn be the space of nxn complex matrices of order n.

For $A \in C_{nxn}$, let $A^T$, $\overline{A}$, $A^*$, $A^S$ and $A^\theta$ denote the transpose, Conjugate, Conjugate transpose secondary transpose and conjugate secondary transpose of matrix A respectively. Throughout, let 'k' be a fixed product of disjoint transpositions in $S_n$ the set of all permutations on {1, 2, 3,…n} and K be the associated permutation matrix with units in the secondary diagonal. 'K' and 'V' clearly satisfies the following properties.

$\overline{K} = K^T = K^S = K^* = \overline{K^S} = K$;  $K^2 = I$

$\overline{V} = V^T = V^S = V^* = \overline{V^S} = V$;  $V^2 = I$

***Definition (2.1): [4]***

A matrix $A \in C_{nxn}$ is said to be secondary normal (s-normal) if $AA^\theta = A^\theta A$, that is an s-normal matrix is one which commutes with its conjugate secondary transpose.

***Definition (2.2): [5]***

A matrix $A \in C_{nxn}$ is said to be k-normal if $AA^*K = KA^*A$.

### *3. s-k normal matrices*

In this section, the concept of s-k normal matrices is introduced.

Definition (3.1):

A matrix $A \in C_{nxn}$ is said to secondary k-normal (s-knormal) matrix if

$$A(KVA^*VK) = (KVA^*VK)A.$$

Example (3.2):

$$A = \begin{pmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{pmatrix}$$ is a s-k normal matrix.

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**Remark (3.3):**

The concept of s-k normal matrices is analogous to that of normal matrices

**Theorem (3.4):**

(i) The transpose of an s-k normal matrix is s-k normal.

(ii) The secondary transpose of an s-k normal matrix is s-k normal.

(iii) The conjugate of an s-k normal matrix is s-k normal.

**Proof:**

Let $A \in C_{n\text{x}n}$.

(i) Since A is s-k normal,

$$A(KVA^{*}VK) = (KVA^{*}VK)A$$

$$(A(KVA^{*}VK))^{T} = ((KVA^{*}VK)A)^{T}$$

$$(KV(A^{*})^{T}VK)A^{T} = A^{T}(KV(A^{*})^{T}VK)$$

$$KV(A^{T})^{*}VKA^{T} = A^{T}KV(A^{T})^{*}VK$$

Therefore $A^T$ is s-k normal

(ii)

$$A(KVA^{*}VK) = (KVA^{*}VK)A$$

$$(A(KVA^{*}VK))^{S} = ((KVA^{*}VK)A)^{S}$$

$$(KV(A^{S})^{*}VK)A^{S} = A^{S}(KV(A^{S})^{*}VK)$$

Thus $A^S$ is s-k normal.

(iii)

$$A(KVA^{*}VK) = (KVA^{*}VK)A$$

$$\overline{A(KVA^{*}VK)} = \overline{(KVA^{*}VK)A}$$

$$\overline{A}(KV(\overline{A})^{*}VK)A = (KV(\overline{A})^{*}VK)\overline{A}$$

Hence $\overline{A}$ is s-k normal

***Theorem (3.5):***

(i) Real secondary k-symmetric matrices are s-k normal.

(ii) Real secondary k-skew symmetric matrices are s-k normal.

(iii) Real secondary k-orthogonal matrices are s-k normal

(iv) secondary k-hermitian matrices are s-k normal.

(v) secondary k-skew hermitian matrices are s-k normal

***Proof***

Let $A \in C_{n\text{x}n}$.

(i) Let A be a real s-k symmetric matrix

Thus $A = KVA^{T}VK = KVA^{*}VK$

which implies that

$$A(KVA^{*}VK) = (KVA^{*}VK)A$$

Therefore A is s-k normal

(ii) If A is a real s-k skew symmetric matrix then

$$A = -(KVA^{T}VK) = -(KVA^{*}VK)$$

$$A(KVA^{*}VK) = (KVA^{*}VK)A.$$

Hence A is s-k normal.

(iii) Let A be a real s-k orthogonal matrix

Then

$A(KVA^{T}VK) = (KVA^{T}VK)A = I$ which leads to

$$A^{-1} = KVA^{T}VK.$$

Since A is real,

$A(KVA^{T}VK) = AA^{-1} = I$ and

$$(KVA^{T}VK)A = A^{-1}A = I$$

Thus A is s-k normal.

(iv) If A is an s-k hermitian matrix, then

$KVA^{*}VK = A$ which implies

$$A(KVA^{*}VK) = A^{2}$$

Also $(KVA^{*}VK)A = A^{2}$

Therefore

$$A(KVA^{*}VK) = (KVA^{*}VK)A.$$

Hence A is s-k normal.

(v) Let A be an s-k skew hermitian matrix

By definition, $KVA^{*}VK = -A$.

$\Rightarrow A(KVA^{*}VK) = -A^{2}$

Also $(KVA^{*}VK)A = -A^{2}$

$\Rightarrow$

$$A(KVA^{*}VK) = (KVA^{*}VK)A.$$

$\Rightarrow$ A is s-k normal.

***Lemma: (3.6):***

Let $A, N \in C_{n\text{x}n}$. If N is s-k normal such that AN=NA and KVNVK=N

then $A(KVN^{*}VK) = (KVN^{*}VK)A$.

***Proof:***

Let $A, N \in C_{n\text{x}n}$ and N be s-k normal.

Let AN=NA and KVNVK=N. Since N commutes with A and $KVN^{*}VK$, it must commute with $A(KVN^{*}VK) - (KVN^{*}VK)A$. For

$$N(A(KVN^{*}VK) - (KVN^{*}VK)A) = NA(KVN^{*}VK) - N(KVN^{*}VK)A$$

$$= AN(KVN^*VK) - (KVN^*VK)NA$$
$$= A(KVN^*VK)N - (KVN^*VK)AN$$

$$N(A(KVN^*VK) - (KVN^*VK)A = (A(KVN^*VK) - (KVN^*VK)A)N$$

Now let

$$X = A(KVN^*VK) - (KVN^*VK)A$$

$$XX^* = (A(KVN^*VK) - (KVN^*VK)A)(A(KVN^*VK) - (KVN^*VK)A)^*$$

$$= (A(KVN^*VK) - (KVN^*VK)A)((KVNVK)A^*)$$

$$(A(KVN^*VK) - (KVN^*VK)A)(A^*(KVNVK))$$
$$= KVNVK(A(KVN^*VK) - (KVN^*VK)A)A^*) -$$
$$(A(KVN^*VK) - (KVN^*VK)A)(A^*(KVNVK)$$
$$= N(A(KVN^*VK) - (KVN^*VK)A)A^*$$
$$-(A(KVN^*VK) - (KVN^*VK)A)A^*)N$$

$XX^* = NB - BN$ where

$$B = (A(KVN^*VK) - (KVN^*VK)A)A^*)$$

⇨ $tr(XX^*) = tr(NB-BN)$
$= tr(NB) - tr(BN)$

⇨ $tr(XX^*) = 0$
$X = 0.$

Hence

$$A(KVN^*VK) = (KVN^*VK)A$$

***Theorem (3.7):***

If A and B are two s-k normal matrices such that AB = BA,

$A(KVB^*VK) = (KVB^*VK)A$ and

$(KVA^*VK)B = B(KVA^*VK)$ then A+B and AB are also s-k normal matrices.

***Proof:***

Let $A, B \in C_{nxn}$.

Let A and B be s-k normal matrices such that AB = BA.

Then by lemma (3.6) $A(KVB^*VK) = (KVB^*VK)A$

and $(KVA^*VK)B = B(KVA^*VK)$

Now,

$$(A+B)\ (KV(A+B)^*VK) = (A+B)(KV\left(A^* + B^*\right)VK)$$
$$= (KV(A+B)^*VK) = (A+B)(KV\left(A^* + B^*\right)VK)$$
$$= (KVA^*VK)A + (KVB^*VK)A + (KVA^*VK)B + (KVB^*VK)B$$
$$= (KV(A^* + B^*)VK)A + (KV(A^* + B^*)VK)B$$
$$= \left(KV(A^* + B^*)VK\right)(A+B)$$
$$= \left(KV(A+B)^*VK\right)(A+B)$$

⇒ A+B is s-k normal.

Now,

$$(AB)(KV(AB)^*VK) = AB(KVB^*A^*VK)$$
$$= AB(KVB^*VK)(KVA^*VK)$$
$$= A(KVB^*VK)B(KVA^*VK)$$
$$= (KVB^*VK)A(KVA^*VK)B.$$
$$= (KVB^*VK)(KVA^*VK)AB.$$
$$= (KVB^*A^*VK)AB$$
$$(AB)(KV(AB)^*VK) = (KV(AB)^*VK)AB.$$

Hence the theorem.

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