NORMAL, REGULARITY AND NEIGHBOURHOOD IN GENERALISED $\beta^*$ CLOSED MAPS

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ARTICLE INFO

Article History:
Received 28th September, 2012
Received in revised form 10th, October, 2012
Accepted 15th October, 2012
Published online 29th November, 2012

Key Words:
m-Structure, beta* set, beta* continous, g-closed to be corrected

INTRODUCTION

Levine [9] introduced the concept of $g$-closed sets and studied their properties. A subset A of a space X is g-closed if and only if $cl(A) \subseteq O$ whenever $A \subseteq O$ and O is open. Hence every closed set is a g-closed set. The union and intersection of two g-closed set is g-closed set. Regular open sets and stronger regular open sets have been introduced and investigated by Stone [19] and Tang [21] respectively. Complements of regular open sets and strong regular open sets are called regular closed sets and strong regular closed sets. Andrijevic[1], Arya and Nour[2], Bhattacharya and Lahiri[5,Levine[9],[10],Masbour et al[13] and Njastad[17] introduced and investigated semi-preopen sets, generalized semi open sets, semi generalized open sets, generalized open sets, semi-open sets, pre-open sets, generalized open set, semi-open sets pre-open sets and $\alpha$-open sets which are some of the weak forms of open sets and the complement of these sets are called the same types of closed sets respectively. Ganster and Reilly [8] have introduced locally closed sets which are weaker than both open and closed sets. Cameron [6] has introduced regular semi-open sets which are weaker than regular open sets. $\beta^*$ open sets and $\beta^*$ continuous functions were already introduced by Palanimani and Parimelazhagan, further the closed maps were studied.

PRELIMINARIES

In this section we begin by recalling some definitions and properties Let $(X, \tau)$ be a topological spaces and A be a subset. The closure of A and interior of A are denoted by $cl(A)$ and $int(A)$ respectively. We recall some generalized open sets.

Definition [9] 2.1: A subset A of a space X is g-closed if and only if $cl(A) \subseteq G$ whenever $A \subseteq G$ and G is open.

Definition [20]2.2: A map $f : X \rightarrow Y$ is called g-closed if each closed set F of X, $f(F)$ is g-closed in Y.

Definition [18]2.3: A map $f : X \rightarrow Y$ is called semi-closed if each closed set F of X, $f(F)$ is semi-closed in Y.

Definition [15] 2.4 : A map $f : X \rightarrow Y$ is called $\alpha$-open if each open set F of X, $f(F)$ is $\alpha$-set in Y.

Definition [7]2.5 : A map $f : X \rightarrow Y$ is called pre-closed if for each closed map F of X, $f(F)$ is pre-closed in Y.

Definition [12]2.6: A map $f : X \rightarrow Y$ is called regular-closed if for each set F of X, $f(F)$ is regular closed in Y.

Definition (11)2.7: A map $f : X \rightarrow Y$ is said to be strongly continuous if $f^{-1}(V)$ is both open and closed in X for each subset V of Y.

Definition [4] 2.8: A map f: X → Y is said to be generalized continuous if $f^{-1}(V)$ is g-open in X for each subset V of Y.

Definition [15] 2.9: A subset A of a topological space X is said to be $\beta^*$ closed set in X if $cl(int(A))$ contained in U whenever U is G-open.

Remark 2.11: The following implications were well known...
3. Properties of $\beta^*$-closed sets

In this section we study some of the properties of $\beta^*$-closed set.

**Definition 3.1:** A map $f: X \to Y$ is called $\beta^*$-closed map if for each closed set $F$ of $X$, $f(F)$ is $\beta^*$-closed set.

**Remark 3.2:** Every $g$-closed map is a $\beta^*$-closed map and the converse is need not be true from the following example.

**Example 3.3:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\tau^c = \{\phi, X, \{b, c\}, \{c\}\}$ be topologies on $X$. $f: X \to Y$ each closed set $f(F)$ is $g$-closed. Here $\{a, c\}$ is g-closed but not $\beta^*$-closed.

**Theorem 3.4:** A map $f: X \to Y$ is $\beta^*$-closed if and only if for each subset $S$ of $Y$ and for each open set $U$ containing $f^{-1}(S)$ there is a $\beta^*$-open set $V$ of $Y$ such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

**Proof:** Suppose $f$ is $\beta^*$-closed. Let $S$ be a subset of $Y$ and $U$ is an open set of $X$ such that $f^{-1}(S) \subseteq U$, then $V = Y \setminus f^{-1}(X \setminus U)$ is a $\beta^*$-open set of $V$ of $Y$ such that $S \subseteq V$ such that $f^{-1}(V) \subseteq U$.

For the converse suppose that $f$ is a closed set of $X$. Then $f^{-1}(Y \setminus f(F)) \subseteq X \setminus F$ and $X \setminus F$ is open. By hypothesis there is $\beta^*$-open set $V$ of $Y$ such that $Y \setminus f(F) \subseteq V$ and $f^{-1}(V) \subseteq X \setminus F$. Therefore $F \subseteq X \setminus f^{-1}(V) \subseteq \text{int}(f^{-1}(V)) \subseteq Y \setminus V$ which implies $f(F) = Y \setminus V$. Since $Y \setminus V$ is $\beta^*$-closed if $f(F)$ is $\beta^*$-closed and thus $f$ is a $\beta^*$-closed map.

**Theorem 3.5:** If $f: X \to Y$ is continuous and $\beta^*$-closed and $A$ is a $\beta^*$-closed set of $X$ then $f(A)$ is $\beta^*$-closed.

**Proof:** Let $A$ be a $\beta^*$-closed set of $X$. Since $f$ is $g$-continuous, $f^{-1}(O)$ is an open set containing $A$. Hence $\text{cl}(\text{int}(A)) \subseteq f^{-1}(O)$ is an $\beta^*$-closed set. Since $f$ is $\beta^*$-closed, $f(\text{cl}(\text{int}(A)))$ is a $\beta^*$-closed set contained in the open set $O$ which implies that $\text{cl}(\text{int}(f(\text{cl}(\text{int}(A))))) \subseteq O$ and hence $\text{cl}(\text{int}(f(\text{cl}(\text{int}(A))))) \subseteq O\cdot f$ is a $\beta^*$-closed set.

**Corollary 3.6:** If $f: X \to Y$ is $g$-continuous and closed and $A$ is $\beta^*$-closed set of $X$ the $f(A)$ is $\beta^*$-closed.

**Corollary 3.7:** If $f: X \to Y$ is $\beta^*$-closed and continuous and $A$ is $\beta^*$-closed set of $X$ then $f(A): A \to Y$ is $\beta^*$-closed.

**Proof:** Let $F$ be a closed set of $A$ then $F$ is $\beta^*$-closed set of $X$. From above theorem 3.5 follows that $f_0(F) = f(F)$ is $\beta^*$-closed set of $Y$. Here $f_0$ is $\beta^*$-closed and continuous.

**Theorem 3.8:** If a map $f: X \to Y$ is closed and a map $g: Y \to Z$ is $\beta^*$-closed then $f: X \to Z$ is $\beta^*$-closed.

**Proof:** Let $H$ be a closed set in $X$. Then $f(H)$ is closed and $(g \circ f)(H) = g(f(H))$ is $\beta^*$-closed as $g$ is $\beta^*$-closed. Thus $g \circ f$ is $\beta^*$-closed.

**Theorem 3.9:** If $f: X \to Y$ is continuous and $\beta^*$-closed and $A$ is a $\beta^*$-closed set of $X$ then $f_A: A \to Y$ is continuous and $\beta^*$-closed.

**Proof:** If $F$ is a closed set of $A$ then $F$ is a $\beta^*$-closed set of $X$. From Theorem 3.4, it follows that $f_0(F) = f(F)$ is a $\beta^*$-closed set of $Y$. Hence $f_A$ is $\beta^*$-closed. Also $f_A$ is continuous.

**Theorem 4.1:** If $f: X \to Y$ is $\beta^*$-closed and $A = f^{-1}(B)$ for some closed set $B$ of $Y$ then $f_0(A \to Y)$ is $\beta^*$-closed.

**Proof:** Let $F$ be a closed set in $A$. Then there is a closed set $H$ in $X$ such that $F = A \cap H$. Then $f_0(F) = f(A \cap H) = f(H) \cap f(B)$. Since $f$ is $\beta^*$-closed, $f(H)$ is $\beta^*$-closed in $Y$. So $f(H) \cap f(B)$ is $\beta^*$-closed in $Y$. Since the intersection of a $\beta^*$-closed and a closed set is a $\beta^*$-closed set. Hence $f_A$ is $\beta^*$-closed.

**Remark 3.11:** If $B$ is not closed in $Y$ then the above theorem does not hold from the following example.

**Example 3.12:** Take $B = \{b,c\}$. Then $A = f^{-1}(B) = \{b, c\}$ and $\{c\}$ is closed in $A$ but $f_0(\{b\}) = \{b\}$ is not $\beta^*$-closed in $Y$. $\{a\}$ is also not $\beta^*$-closed in $B$.

### 4. Normal and Regularity

In this section we introduce the new class of $\beta^*$-regular and studied some of its properties.

**Theorem 4.1:** If $f: X \to Y$ is continuous, $\beta^*$-closed map from a normal space $X$ onto a space $Y$ then $Y$ is normal.

**Proof:** Let $A, B$ be disjoint closed sets in $Y$. Then $f^{-1}(A), f^{-1}(B)$ are disjoint closed sets of $X$. Since $X$ is normal, there are disjoint open sets $U, V$ in $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since $f$ is $\beta^*$-closed by theorem 3.4, there are $\beta^*$-open sets $G, H$ in $Y$ such that $A \subseteq G, B \subseteq H$ and $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since $U, V$ are disjoint, $\text{int} G, \text{int} H$ are disjoint open sets. Since $G$ is $\beta^*$-open, $A$ is closed and $A \subseteq G \subseteq \text{cl}(\text{int}(G))$. Similarly $B \subseteq \text{cl}(\text{int}(H))$. Hence $Y$ is normal.

**Theorem 4.2:** If $f: X \to Y$ is an open continuous $\beta^*$-closed surjection, where $X$ is regular then $Y$ is regular.

**Proof:** Let $U$ be an open set containing a point $P$ in $Y$. Let $X$ be a point of $X$ such that $f(X) = P$. Since $X$ is regular and $f$ is continuous there is an open set $U$ such that $x \in V \subseteq \text{cl}(V) \subseteq f^{-1}(V)$. Hence $P \in f(V) \subseteq f(\text{cl}(V)) \subseteq U$. Since $f$ is $\beta^*$ closed, $f(\text{cl}(V))$ is $\beta^*-$
closed set contained in the open set U. It follows that cl(int(f(cl(int(V)))))) ∈ U and hence p ∈ cl(V) ∩ cl(int(V)) ⊆ U and f(V) is open. Since f is open. Hence Y is regular.

Remark 4.3: The normality is preserved under regular closed, continuous and surjective.

Example 4.4: In the example 3.12. It is shown that f is a β*-closed [b,c] is a regular closed set in (X, τ₁) and it is not closed in (X, τ₂). Hence f is not regular closed.

Example 4.5: Let τᵢ be the countable complement topology on the real line R and τ₂ be the usual topology on R and f : (R, τ₁) → (R, τ₂) be the identity map. Then f is closed by the remark immediately after the above example. But f is not β*-closed. If A = {1/n, n ∈ N} then A is closed in (R, τ₁) and f(A) = A is not β*-closed as f(A) ⊆ (0, 2) and (0, 2) is open in (R, τ₂). But cl(int(f(A))) ⊆ (0, 2).

Theorem 4.6: If A is a β*-closed set of a space X then Ind A ≤ Ind X

Proof: It suffices to show that if Ind X ≤ n and A is a β*-closed set of X then Ind A ≤ n. We prove this theorem by induction. The result holds trivially for n = 1. Assume that for every β*-closed set A of X, ind X ≤ n − 1 ⇒ Ind A ≤ n − 1.

Let X be space with Ind ≤ n. Let A be a β*-closed set of X. Let E be a closed set of A and G be an open set of A such that E ⊆ G. Then there exist a closed set F of X and an open set H of X such that F = A ∩ F and G = A ∩ H. Since E is closed in A and A is β*-closed. Since Ind X ≤ n, there is an open set V of X such that cl(int(E)) ⊆ V ⊆ H and Indbd(V) ≤ n − 1. Then V ∩ A is an open set of A such that E ⊆ V ∩ A ⊆ G and bd(J ∩ A) ⊆ bd(V). Now bd(J ∩ A) is a β*-closed set of bd(V). By induction hypothesis and Indbd(β) ∩ A) ≤ n − 1. Hence Ind A ≤ n.

Theorem 4.7: If A is a β*-closed set of a space X then dim A ≤ dim X.

Proof: If dim X = 0 then dim A ≤ 0 = dim X. Hence dim A ≤ dim X. If dim X = 0 then dim X = 0, where n is an integer greater than or equal to -1. If n = -1 dim X = -1 which implies that X = φ and hence A = φ and dim A = -1 = dim X and thus dim A ≤ dim X.

Next suppose dim X = n where n ≥ -1 and let A be a β*-closed set of X. Let {u₁, u₂, u₃, ..., uₖ} be a finite open cover of A. Then for i = 1, 2, 3, ...K there exist open sets V_i of X such that u_i = A ∩ V_i. Since A is a β*-closed and U_{β} = V_i is an open set containing A, cl(int(A)) ⊆ U_{β} = V_i. Since cl(int(A)) is a closed set, dim cl(int(A)) ≤ n, so the finite open cover {cl(int(A ∩ V_i) i = 1, 2, 3, ...k} cl(int(A)) has a refinement cl(int(A)) ⊆ w_i, i = 1, 2, 3, ...k or order at most n + 1, where each w_i is open in X and cl(int(A)) ∩ w_i ⊆ cl(int(A)) ∩ V_i for each i. Then {A ∩ V_i : i = 1, 2, ...} is an open cover of A refining {u_i, i = 1, 2, 3, ...k} and of order not exceeding n + 1. Hence dim A ≤ n which implies that dim A ≤ dim X.

Theorem 4.8: If A is a β*-closed set of a space X then Dind A ≤ Dind X.

Proof: Let X be a space such that Dind X = n and A be a β*-closed set of X. By using the notations of the above theorem, cl(int(A)) ⊆ U_i. Since cl(int(A)) is a closed set, Dind A ≤ n. Hence for every open cover V_i ∩ cl(int(A)), i = 1, 2, 3, ...k there is a disjoint family W_i, j = 1, 2, 3, ...k of open sets cl(int(A)) refining V_i ∩ cl(int(A)), i = 1, 2, 3, ...k and such that Dind(cl(int(A)) - U_{β} = V_i) ≤ n. But A - U_{β} = V_i ⊆ cl(int(A)) - U_{β} = V_i and A - U_{β} = V_i = A ∩ (cl(int(A)) - U_{β} = V_i) is a β*-closed set of (cl(int(A)) - U_{β} = V_i) as the intersection of β*-closed sets. By induction hypothesis Dind(A) - U_{β} = V_i ≤ n - 1. Also W_i ∩ A, j = 1, 2, 3, ...k is a disjoint family of open sets of A refining u_1, u_2, u_3, ..., u_k. Thus Dind A ≤ n and the theorem is proved.

5. β*-Open sets and β*-Neighborhoods

In this section we introduce β*-neighborhoods (β*-nbhd) topological spaces by using the notion of β*-open sets and study some properties.

Definition 5.1: Let X be the point in topological space X, then the set of all β*-neighborhood of a X is called β*-nbhd system of X which is denoted by β*-N(X).

Theorem 5.2: Let X be the topological space and each x ∈ X. Let β*-N(x,τ) be the collection of all β*-nbhd of X then we have the following results

(i) ∀ x ∈ X, β*- N(X) ≠ φ
(ii) N ∈ β*- N(X) ⇒ x ∈ N
(iii) N ∈ β*- N(X), M ⊆ N ⇒ M ∈ β*- N(X)
(iv) N ∈ β*- N(X) ⇒ ∃ M ∈ β*- N(X) such that M ⊆ N and M ∈ β*- N(Y), ∀ Y ∈ M

Proof: (i) Since X is β*-open set, it is β*-nbhd of every x ∈ X, Hence there exists at least one β*-nbhd (namely X) for each x ∈ X. Hence ∀ x ∈ X, β*- N(X) ≠ φ

(ii) if N ∈ β*- N(X), then N is a β*-nbhd of x, then by definition β*-nbhd(x) ∈ N
(iii) Let $N \in \beta^*-\text{nbd}$ and $M \supseteq N$, then there is a 
$\beta^*$-open set $U$ such that $x \in U \subseteq N$

Since $N \subseteq M$, $x \in U \subseteq M$ and $M$ is $\beta^*$-nbd of $X$.

Hence $M \in \beta^* - N(X)$

(iv) If $N \in \beta^* - N(X)$, then there exists a $\beta^*$-open set such that $x \in M \subseteq N$, since $M$ is a $\beta^*$-open set, it is $\beta^*$-nbd of each of its points.

Therefore $M \in \beta^* - N(Y)$ for every $Y \in M$

**Theorem 5.3.** Let $X$ be a nonempty set, for each $x \in X$, let $\beta^*-N(x)$ be nonempty collection of subsets of $X$ satisfying following conditions.

(i) $N \in \beta^*-N(X, \tau ) \Rightarrow x \in N$.

(ii) Let $\tau$ consists of the empty set and all those non-empty subsets of $U$ of $X$ having the property that $x \in U$ implies that there exists an $N \subseteq \beta^*-N(X)$ such that $x \in N \subseteq U$. Then $\tau$ is a topology for $X$.

**Proof:** (i) $\phi \in \tau$ by definition. We now show that $x \in \tau$.

Let $x$ be any arbitrary element of $X$. Since $\beta^*-N(x)$ is non-empty, there is an $N \in \beta^*-N(x)$ and so $x \in N$.

Since $N$ is a subset of $X$, we have $x \in N \subseteq X$.

Hence $X \in \tau$.

(ii) Let $U_{\lambda} \in \tau$ for every $\lambda \in \Lambda$. If $x \in U \{U_{\lambda} : \lambda \in \Lambda \}$, then $x \in U_{\lambda x}$ for some $\lambda x \in \Lambda$.

Since $U_{\lambda x} \in \tau$, there exists an $N \in \beta^*-N(x)$ such that $x \in N \subseteq U_{\lambda x}$ and consequently $x \in N \subseteq U \{U_{\lambda} : \lambda \in \Lambda \}$. Hence $U \{U_{\lambda} : \lambda \in \Lambda \} \in \tau$. It follows that $\tau$ is topology for $X$.

**References**


