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Research Article

# COINCIDENCE AND COMMON FIXED POINTS RESULTS FOR HYBRID PAIRS OF MAPPINGS WITH SOME PROPERTIES

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**ABSTRACT**

The aim of this paper is to obtain some coincidence and common fixed point theorems for hybrid pairs of mappings in b-metric spaces by using the ($CLR_f$)-property and the (owc)-property. Examples are given to indicate the usefulness of our main results. These results improve, extend and generalize the corresponding results in the literature.



## 1. INTRODUCTION

The idea of metric space was introduced by Frechet in 1906, is one of the cornerstones of not only mathematics but also several quantitative sciences. Due to its importance and application potential, this idea has been extended, improved and generalized in many different ways. In this paper, we focus towards the concept of b-metric spaces. In 1993, Czerwik [16] introduced the notion of b-metric spaces which generalized the concept of metric spaces. Many authors worked on fixed points of multivalued mappings in different directions in these spaces (for more details see [3], [4], [11], [14]).

The concept of compatibility, which was introduced by Jungck [5] for single valued mappings in metric spaces, has been extended to multivalued mappings by Kaneko and Sessa [10] with Hausdorff distance. In 1996, Jungck [6] defined the notion of weakly compatible mappings in metric spaces and proved some common fixed point theorems for such mappings. We can consider [7] and [9] to illustrate the relation that compatible mappings are weakly compatible, but converse is not true. Jungck and Rhoades [8] in 2006, coined the idea of occasionally weakly compatible mappings ((owc)- property). By introducing the notion of (owc)- property, Abbas and Rhoades [12] generalized the concept of weakly compatible mappings in setting of single and multivalued mappings. In 2009, Aliouche and Popa [1] proved some common fixed point theorems for hybrid pair of mappings in symmetric spaces using (owc)- property.

Aamri and Moutawakil [13] in 2002, defined the idea of (E. A) property for self mappings which contained the class of non-compatible mappings in metric spaces. The (E. A) property requires the completness (closedness) for the existence of the fixed point in the underlying subspace. To relaxes the requirement of completness (closedness), very first common limit range property with respect to mapping f (($CLR_f$)-property) is introduced by Sintunaravat and Kumam [21] regarding fuzzy metric space after that this property is used in many other spaces which showed the superiority of ($CLR_f$)-property than (E. A) property.

Recently, Noan Abdou [2] introduced the notion of (owc)-property and ($CLR_f$)-property for four single valued and multivalued mappings in metric space and proved some coincidence and common fixed point theorems. Motivated by [2] firstly, we obtain some common fixed point theorems for hybrid mappings along with the (owc)-property using a symmetric $\delta$ derived from an ordinary symmetric d. Secondly, we prove common and coincidence fixed point theorems for

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hybrid mappings along with ($CLR_f$)- property in the setting of complete b-metric spaces.

***2. Mathematical Preliminaries***

The following are the concepts from set valued analysis which we shall use in this paper. Let (X, d) be a metric space. Then

B(X) = {A: A is a non-empty bounded subset of X},
CL(X) = {A: A is a non-empty closed subset of X} and
CB(X) = {A: A is a non-empty closed and bounded subset of X}.

Let T be a multivalued mapping of X in to CB(X) and f be a self mapping of X. Then the pair (f, S) is said to be a hybrid pair. An element $x \in X$ is said to be a coincidence point of $T : X \to CB(X)$ and $f : X \to X$ if $fx \in Tx$. We denote $C(f, T) = \{x \in X : fx \in Tx\}$, the set of coincidence point of T and f.

***Definition 2.1:-*** Let f, g : X → X be two self mappings. Then the pair (f, g) is said to

1. be compatible [5] if $\lim_{n\to+\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$;
2. be non-compatible if there is at least one sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$, but $\lim_{n\to+\infty} d(fgx_n, gfx_n)$ is either nonzero or nonexistent;
3. be weakly compatible if fgx = gfx whenever fx = gx, $x \in X$;
4. be occasionally weakly compatible (owc) [8] if fgx = gfx for some $x \in C(f, g)$;
5. satisfy the property (E. A) [13] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$, for some $t \in X$;
6. satisfy common limit range property with respect to the mapping f ($CLR_f$) [21] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = fu$, for some $u \in X$.

***Definition 2.2:-*** Let f : X → X and T : X → CB(X) be a single valued and multivalued mapping respectively. Then a hybrid pair of mappings (f, T) is said to

1. be compatible [10] if $fTx \in CB(X)$ for all $x \in X$ and $\lim_{n\to+\infty} H(Tfx_n, fTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that
$$\lim_{n\to+\infty} Tx_n \to A \in CB(X) \text{ and } \lim_{n\to+\infty} fx_n \to t \in A;$$
2. be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that
$$\lim_{n\to+\infty} Tx_n \to A \in CB(X) \text{ and } \lim_{n\to+\infty} fx_n \to t \in A$$
but $\lim_{n\to+\infty} H(Tfx_n, fTx_n)$ is either nonzero or nonexistent;
3. be weakly compatible if Tfx = fTx whenever $fx \in Tx$;
4. be occasionally weakly compatible (owc) [12] if and only if there exists some point $x \in X$ such that $fx \in Tx$ and $fTx \subset Tfx$;
5. satisfy the property (E. A) [20] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} fx_n = t \in A = \lim_{n\to+\infty} Tx_n$ for some $t \in X$ and $A \in CB(X)$;
6. satisfy common limit range property with respect to the mapping f ($CLR_f$) [15] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} fx_n = fu \in A = \lim_{n\to+\infty} Tx_n$, for some $u \in X$ and $A \in CB(X)$.

***Remark 2.3:-***

1. Every pair of non-compatible self mappings of a metric space (X, d) satisfies property (E. A), but its converse need not be true.
2. Every compatible pair is weakly compatible but its converse need not be true.
3. Every weakly compatible pair is occasionally weakly compatible but its converse need not be true.

***Definition 2.4[16]:-*** Let X be a non empty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1. d(x, y) = 0 if and only if x = y;
2. d(x, y) = d(y, x) for all $x, y \in X$;
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then (X, d, s) is called a b-metric space.

Note that a (usual) metric space is evidently b-metric space. However, Czerwik [16, 17, 18] has shown that a b-metric on X need not be a metric on X. In following example, Singh and Prasad [19] proved that a b-metric on X need not be a metric on X.

***Example 2.5[19]:-*** Consider the set X = [0, 1] endowed with the function $d : X \times X \to R^+$ defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Clearly, (X, d) is a b-metric space with s = 2, but it is not a metric space.

***Example 2.6[19]:-*** Let X = {a, b, c} and d(a, c) = d(c, a) = $m \geq 2$, d(a, b) = d(b, c) = d(b, a) = d(c, b) = 1 and d(a, a) = d(b, b) = d(c, c) = 0. Then,

$$d(x, y) \leq \frac{m}{2}[d(x, z) + d(z, y)]$$

for all $x, y, z \in X$. If m > 2, the triangle inequality does not hold.

***Definition 2.7[16]:-*** Let (X, d, s) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:

1. Convergent if and only if there exist $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.
2. Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
3. Complete if and only if every Cauchy sequence is convergent.

Let (X, d, s) be a b-metric space. For $A, B \in CB(X)$ and $x \in X$, define the function $H: CB(X) \times CB(X) \to R^+$ by

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

where $\delta(A, B) = \sup\{d(a, B) : a \in A\}$,
$\delta(B, A) = \sup\{d(b, A) : b \in B\}$ and

$d(x, A) = \inf\{d(x, a), a \in A\}$.

Note that H is called the Hausdorff b-metric induced by the b-metric d.

***Remark 2.8[16]:-*** The function $H: CL(X) \times CL(X) \to R^+$ is a generalized Hausdorff b-metric, that is $H(A, B) = +\infty$ if $\max\{\delta(A, B), \delta(B, A)\}$ do not exist.

Let (X, d, s) be a b-metric space. We cite the following lemma from Singh and Prasad [19].

***Lemma 2.9[19]:-*** Let (X, d, s) be a b-metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$, we have the following:

1. $d(x, B) \leq d(x, b)$ for any $b \in B$,
2. $d(x, B) \leq H(A, B)$ for all $x \in A$,
3. $\delta(A, B) \leq H(A, B)$,
4. $H(A, A) = 0$,
5. $H(A, B) = H(B, A)$,
6. $H(A, C) \leq s(H(A, B) + H(B, C))$,
7. $d(x, A) \leq s(d(x, y) + d(y, A))$.

To prove our results we need the following class of functions. Let $s \geq 1$ be a real number, we denote $\Psi_s$ the family of continuous monotone increasing functions in b-metric space, $\varphi : [0, \infty) \to [0, \infty)$ such that

$$\sum_{n=0}^{\infty} s^n \varphi^n(t) < +\infty \text{ for each } t > 0,$$

where $\varphi^n$ denotes n-th iterate of the function φ. It is well known that $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$ for $t = 0$. An example of function $\varphi \in \Psi_s$ is given by $\varphi(t) = \frac{ct}{s}$ for all $t \geq 0$, where $c \in (0, 1)$.

### *3. Common fixed points for mappings with the (owc)-property*

Now, we prove the main results in this section.

***Theorem 3.1:-*** Let (X, d, s) be a b-metric space. Let $f, g : X \to X$ be single-valued mappings and $S, T : X \to B(X)$ be multi-valued mappings satisfying the following conditions:

1. the pair (S, f) and (T, g) are the (owc)-property,
2. for all $x, y \in X$,

$$\delta(Sx, Ty) \leq \varphi\Big(\max\Big\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2s}[d(fx, Ty) + d(gy, Sx)], \frac{d(fx,Sx)(1+d(gy,Ty))}{1+d(fx,gy)}, \frac{d(fx,Ty)(1+d(gy,Sx))}{1+d(fx,gy)}\Big\}\Big).$$

Then f, g, S and T have a unique common fixed point in X.

***Proof:-*** Since the pairs (S, f) and (T, g) satisfy the (owc)-property, there exist $u, v \in X$ such that

$$fu \in Su, \quad fSu \subset Sfu, \quad gv \in Tv, \quad gTv \subset Tgv,$$

which implies that $ffu \in Sfu$ and $ggv \in Tgv$. Now, we prove that $fu = gv$. In fact, if $fu \neq gv$, then, using the condition (2), we have

$$\delta(Su, Tv) \leq \varphi\Big(\max\Big\{d(fu, gv), d(fu, Su), d(gv, Tv), \frac{1}{2s}[d(fu, Tv) + d(gv, Su)], \frac{d(fu,Su)(1+d(gv,Tv))}{1+d(fu,gv)}, \frac{d(fu,Tv)(1+d(gv,Su))}{1+d(fu,gv)}\Big\}\Big)$$

$$\leq \varphi\Big(\max\Big\{d(fu, gv), \frac{1}{2s}[d(fu, Tv) + d(gv, Su)], \frac{d(fu,Tv)(1+d(gv,Su))}{1+d(fu,gv)}\Big\}\Big).$$

Since $fu \in Su$ and $gv \in Tv$, we have

$$\frac{d(fu,Tv)(1+d(gv,Su))}{1+d(fu,gv)} \leq \frac{d(fu,gv)(1+d(gv,fu))}{1+d(fu,gv)} = d(fu, gv)$$

and $\frac{1}{2s}[d(fu, gv) + d(gv, fu)] = \frac{1}{2s}[2d(fu, gv)] < d(fu, gv)$

and hence

$$\delta(Su, Tv) \leq \varphi(d(fu, gv)).$$

Thus it follows from the property of φ that

$$d(fu, gv) \leq \delta(Su, Tv) \leq \varphi(d(fu, gv)) < d(fu, gv),$$

which is a contradiction and so $fu = gv$.

Next, we prove that fu is a fixed point of f. Suppose that $ffu \neq fu$. Then, by using the condition (2), we have

$$d(ffu, fu) = d(ffu, gv) \leq \delta(Sfu, Tv)$$
$$\leq \varphi\Big(\max\Big\{d(ffu, gv), d(ffu, Sfu), d(gv, Tv), \frac{1}{2s}[d(ffu, Tv) + d(gv, Sfu)], \frac{d(ffu,Sfu)(1+d(gv,Tv))}{1+d(ffu,gv)}, \frac{d(ffu,Tv)(1+d(gv,Sfu))}{1+d(ffu,gv)}\Big\}\Big).$$

Since $ffu \in Sfu$ and $gv \in Tv$, we have

$$\frac{d(ffu,Tv)(1+d(gv,Sfu))}{1+d(ffu,gv)} \leq \frac{d(ffu,gv)(1+d(gv,ffu))}{1+d(ffu,gv)} = d(ffu, gv)$$

and $\frac{1}{2s}[d(ffu, gv) + d(gv, ffu)] = \frac{1}{2s}[2d(ffu, gv)] < d(ffu, gv)$

and hence

$$\delta(Sfu, Tv) \leq \varphi(d(ffu, gv)).$$

Thus it follows from the property of φ that

$$d(ffu, fu) = d(ffu, gv) \leq \delta(Sfu, Tv) \leq \varphi(d(ffu, gv)) < d(ffu, gv) = d(ffu, fu),$$

which is a contradiction and so $ffu = fu$. Similarly, we can prove $fu = gfu = ffu$. Thus we have

$$fu = ffu \in Sfu \text{ and } fu = gfu = ggv \in Tgv = Tfu.$$

Therefore, fu is a common fixed point of f, g, S and T. Moreover, by the condition (2), we have

$$\delta(Sfu, Tfu)$$
$$\leq \varphi\Big(\max\Big\{d(ffu, gfu), d(ffu, Sfu), d(gfu, Tfu), \frac{1}{2s}[d(ffu, Tfu) + d(gfu, Sfu)], \frac{d(ffu,Sfu)(1+d(gfu,Tfu))}{1+d(ffu,gfu)}, \frac{d(ffu,Tfu)(1+d(gfu,Sfu))}{1+d(ffu,gfu)}\Big\}\Big)$$
$$= \varphi(\max\{0, 0, 0, 0, 0\}) = 0.$$

Therefore $Sfu = Tfu = \{fu\}$.

Next, assume that $w \neq z$ is another common fixed point of f, g, S and T. From the condition (2), we have

$$d(z, w) = \delta(Sz, Tw)$$
$$\leq \varphi(\max\{d(fz, gw), d(fz, Sz), d(gw, Tw),$$

$$\frac{1}{2s}[d(fz, Tw) + d(gw, Sz)], \frac{d(fz,Sz)(1+d(gw,Tw))}{1+d(fz,gw)}, \frac{d(fz,Tw)(1+d(gw,Sz))}{1+d(fz,gw)}\})$$

$$= \varphi(\max\{d(z, w), 0, 0, \frac{1}{2s}[d(z, w) + d(w, z)], 0, \frac{d(z,w)(1+d(w,z))}{1+d(w,z)}\})$$

$$= \varphi(d(z, w)) < d(z, w),$$

which is a contraction. Thus the common fixed point z is unique. This completes the proof.

***Example 3.2:-*** Let X = [0, 5], and $d(x, y) = |x - y|^2$ for all x, y ≥ 0. Then (X, d) be a b-metric space with s = 2. Define f, g : X → X and S, T : X → B(X) by

$$Sx = \begin{cases} \{0\} & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \left[\frac{1}{6}, \frac{1}{3}\right] & \text{if } x \in \left(\frac{1}{2}, 5\right], \end{cases}$$

$$fx = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 5 & \text{if } x \in \left(\frac{1}{2}, 5\right], \end{cases}$$

$$Tx = \begin{cases} \{0\} & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \left[\frac{1}{6}, \frac{1}{4}\right] & \text{if } x \in \left(\frac{1}{2}, 5\right], \end{cases}$$

$$gx = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 3 & \text{if } x \in \left(\frac{1}{2}, 5\right], \end{cases}$$

for all x, y ∈ X. Then the pair (S, f) and (T, g) satisfy the (owc)-property because f(0) ∈ S(0), fS(0) ⊆ Sf(0), g(0) ∈ T(0), gT(0) ⊆ Tg(0). Now, we verify that the mappings f, g, S, T satisfy the condition (2) of Theorem 3.1 with $\varphi(t) = \frac{t}{4}$. We have the following cases:

1 If x, y ∈ $\left[0, \frac{1}{2}\right]$, it is obvious.

2 If x ∈ $\left[0, \frac{1}{2}\right]$ and y ∈ $\left(\frac{1}{2}, 5\right]$, we obtain

$$\delta(Sx, Ty) = \left|0 - \frac{1}{6}\right|^2 = \frac{1}{36} \le \frac{9}{4} \le \frac{1}{4} d(fx, gy)$$

$$\le \varphi(\max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2s}[d(fx, Ty) + d(gy, Sx)], \frac{d(fx,Sx)(1+d(gy,Ty))}{1+d(fx,gy)}, \frac{d(fx,Ty)(1+d(gy,Sx))}{1+d(fx,gy)}\})$$

3 If x, y ∈ $\left(\frac{1}{2}, 5\right]$, we obtain

$$\delta(Sx,Ty) = \left|\frac{1}{6} - \frac{1}{3}\right|^2 = \frac{1}{36} \le \frac{36}{4} \le \frac{1}{4}\frac{d(fx,Sx)(1+d(gy,Ty))}{1+d(fx,gy)}$$

$$\le \varphi(\max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2s}[d(fx, Ty) + d(gy, Sx)], \frac{d(fx,Sx)(1+d(gy,Ty))}{1+d(fx,gy)}, \frac{d(fx,Ty)(1+d(gy,Sx))}{1+d(fx,gy)}\}).$$

Therefore, all the conditions of Theorems 3.1 are satisfied and 0 is the common foxed point of mappings f, g, S and T.
If we take S = T and f = g in Theorem 3.1, then we have

***Corollary 3.3:-*** Let (X, d, s) be a b-metric space. Let f : X → X be single-valued mapping and S : X → B(X) be multi-valued mapping satisfying the following conditions:

1. the pair (S, f) satisfies the (owc)-property,
2. for all x, y ∈ X,

$$\delta(Sx, Sy) \le \varphi(\max\{d(fx, fy), d(fx, Sx), d(fy, Sy), \frac{1}{2s}[d(fx, Sy) + d(fy, Sx)], \frac{d(fx,Sx)(1+d(fy,Sy))}{1+d(fx,fy)}, \frac{d(fx,Sy)(1+d(fy,Sx))}{1+d(fx,fy)}\}).$$

Then f and S have a unique common fixed point in X.

***Corollary 3.4:-*** Let (X, d, s) be a b-metric space. Let f, S : X → X be two single-valued mappings satisfying the following conditions:

1. the pair (S, f) satisfies the (owc)-property,
2. for all x, y ∈ X,

$$d(Sx, Sy) \le \varphi(\max\{d(fx, fy), d(fx, Sx), d(fy, Sy), \frac{1}{2s}[d(fx, Sy) + d(fy, Sx)], \frac{d(fx,Sx)(1+d(fy,Sy))}{1+d(fx,fy)}, \frac{d(fx,Sy)(1+d(fy,Sx))}{1+d(fx,fy)}\}).$$

Then f and S have a unique common fixed point in X.

***Corollary 3.5:-*** Let (X, d, s) be a b-metric space. Let f, g, S, T : X → X be four single-valued mappings satisfying the following conditions:

1. the pair (S, f) and (T, g)satisfies the (owc)-property,
2. for all x, y ∈ X,

$$d(Sx, Ty) \le \varphi(\max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2s}[d(fx, Ty) + d(gy, Sx)], \frac{d(fx,Sx)(1+d(gy,Ty))}{1+d(fx,gy)}, \frac{d(fx,Ty)(1+d(gy,Sx))}{1+d(fx,gy)}\}).$$

Then f, g, S and T have a unique common fixed point in X.

If we take φ = kt for some [0, 1) in corollary 3.5, then we have the following.

***Corollary 3.6:-*** Let (X, d, s) be a b-metric space. Let f, g, S, T: X → X be four single-valued mappings satisfying the following conditions:

1. the pair (S, f) and (T, g)satisfies the (owc)-property,
2. for all x, y ∈ X,

$$d(Sx, Ty) \le k\,(\max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2s}[d(fx, Ty) + d(gy, Sx)], \frac{d(fx,Sx)(1+d(gy,Ty))}{1+d(fx,gy)}, \frac{d(fx,Ty)(1+d(gy,Sx))}{1+d(fx,gy)}\}).$$

Then f, g, S and T have a unique common fixed point in X.

### *4.Common fixed points for mappings with the (CLR)-property*

The following is the definition of ($CLR_f$)-property for two hybrid pairs of single-valued and multivalued mappings in metric spaces.

***Definition 4.1[2]:-*** Let (X, d) be a metric space. Two single-valued mappings f, g: X → X and two multivalued mappings S, T : X → CB(X) are said to satisfy the common limit in the range of f (shortly, the ($CLR_f$)-property) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X and A, B ∈ CB(X) such that

$$\lim_{n\to+\infty} Sx_n = A,\quad \lim_{n\to+\infty} Ty_n = B, \text{ and}$$
$$\lim_{n\to+\infty} gx_n = \lim_{n\to+\infty} gy_n = fu \in A \cap B$$

for some u ∈ X.

***Example 4.2[2]:-*** Let X = [1, ∞) with the usual metric. Define two single-valued mappings f , g : X → X and two multivalued mappings S, T : X → CB(X) by

$$fx = 2 + \frac{x}{3},\ gx = 2 + \frac{x}{2},\ Sx = [1, x + 2] \text{ and } Tx = [\,3, 3 + \frac{x}{2}\,],$$

for all x ∈ X, respectively. Then the mappings f and T satisfy the ($CLR_f$)-property for the sequence $\{x_n\}$ and $\{y_n\}$ defined by $x_n = 3 + \frac{1}{n}$ and $y_n = 2 + \frac{1}{n}$ for each n ≥ 1, respectively. Indeed, we have

$$\lim_{n\to+\infty} Sx_n = [1, 5] = A,\quad \lim_{n\to+\infty} Ty_n = [3, 4] = B, \text{ and}$$
$$\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gy_n = 3 = f(3) = A \cap B$$

Therefore, the pairs (S, f) anf (T, g) satisfy the ($CLR_f$)-property.

Now, we prove the main results in this section.

***Theorem 4.3:-*** Let (X, d, s) be a b-metric space. Let f, g : X → X be two single-valued mappings and S, T : X → B(X) be two multi-valued mappings satisfying the following conditions:

1. the pair (S, f) and (T, g) satisfy the ($CLR_f$)-property,
2. for all x, y ∈ X,

$$H(Sx, Ty) \le \varphi\,(\max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2s}[d(fx, Ty) + d(gy, Sx)], \frac{d(fx,Sx)(1+d(gy,Ty))}{1+d(fx,gy)}, \frac{d(fx,Ty)(1+d(gy,Sx))}{1+d(fx,gy)}\}).$$

If f(X) and g(X) are subsets of X, then we have the following:

1. f and S have a coincidence point,
2. g and T have a coincidence point,
3. f and S have a common fixed point provided that f and S are weakly compatible at v and ffv = fv for any v ∈ C(f, S),
4. g and T have a common fixed point provided that g and T are weakly compatible at v and ggv = gv for any v ∈ C(g, T),
5. f, g, S, T have a common point provided that both (3) and (4) are true.

***Proof:-*** Since the pairs (S, f) and (T, g) satisfy the ($CLR_f$)-property, then there exist two sequences$\{x_n\}$and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Sx_n = A,\ \lim_{n\to+\infty} Ty_n = B,\ \lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gy_n = fu \in A \cap B$$

for some u ∈ X. Now, we show that gw ∈ Tw. In fact, suppose that gw ∉ Tw. Then, using the condition (b) with $x = x_n$ and y = w, we have

$$H(Sx_n, Tw) \le \varphi\,(\max\{d(fx_n, gw), d(fx_n, Sx_n), d(gw, Tw), \frac{1}{2s}[d(fx_n, Tw) + d(gw, Sx_n)], \frac{d(fx_n,Sx_n)(1+d(gw,Tw))}{1+d(fx_n,gw)}, \frac{d(fx_n,Tw)(1+d(gw,Sx_n))}{1+d(fx_n,gw)}\})$$

for all n ∈ ℕ. Taking the limit n → ∞, we obtain

$$H(A, Tw) \le \varphi\,(\max\{d(fv, gw), d(fv, A), d(gw, Tw), \frac{1}{2s}[d(fv, Tw) + d(gw, A)], \frac{d(fv,A)(1+d(gw,Tw))}{1+d(fv,gw)}, \frac{d(fv,Tw)(1+d(gw,A))}{1+d(fv,gw)}\})$$
$$= \varphi\,(\max\{0, 0, 0, 0, 0\} = 0.$$

Since gw ∈ A, it follows from the definition of Hausdorff metric that

$$d(gw, Tw) \le H(A, Tw) = 0,$$

which is a contradiction and so gw ∈ Tw. On the other hand, by the condition (b) again, we have

$$H(Sv, Ty_n) \le \varphi\,(\max\{d(fv, gy_n), d(fv, Sv), d(gy_n, Ty_n), \frac{1}{2s}[d(fv, Ty_n) + d(gy_n, Sv)], \frac{d(fv,Sv)(1+d(gy_n,Ty_n))}{1+d(fv,gy_n)}, \frac{d(fv,Ty_n)(1+d(gy_n,Sv))}{1+d(fv,gy_n)}\})$$

for all n ∈ ℕ. Similarly, by taking the limit n → ∞, we obtain

$$H(Sv, B) \le \varphi\,(\max\{d(fv, gw), d(fv, Sv), d(gw, B), \frac{1}{2s}[d(fv, B) + d(gw, Sv)], \frac{d(fv,Sv)(1+d(gw,B))}{1+d(fv,fu)}, \frac{d(fv,B)(1+d(gw,Sv))}{1+d(fv,fu)}\})$$

$= \varphi(\max\{0, 0, 0, 0, 0\} = 0.$

Since $fv \in B$, it follows from the definition of Hausdorff metric that

$d(fv, Sv) \leq H(B, Sv) = 0,$

which is a contradiction and so $fv \in Sv$. Thus the mappings f, S have a coincidence point v and g, T have a coincidence point w. Futhermore, by virtue of the condition (b), we obtain $ffv = fv$ and $ffv \in Sfv$. Thus $u = fu \in Su$. This proves (3). A similar argument proves (4). Thus (5) holds immediately. This completes the proof.

***Example 4.4:-*** Let $X = [1, \infty)$ and $d(x, y) = |x - y|^2$ for all $x, y \geq 0$. Then (X, d) be a b-metric space with s = 2. Define two single valued mappings $f, g : X \to X$ and two multivalued mappings $S, T : X \to B(X)$ by

$fx = gx = x^2$, $Sx = Tx = [1, x + 2]$

for all $x \in X$, respectively. Then the pair (S, f) satisfies the ($CLR_f$)-property with respect to S for the sequence $\{x_n\}$ in X, defined by $x_n = y_n = 1+\frac{1}{n}$ for each $n \geq 1$. Clearly, we have

$$\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gy_n = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^2 = 1 = f(1)$$

and $f(1) \in [1, 3] = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Ty_n$.

Clearly, the pairs f, g, S and T satisfy the condition (b) in Theorem 4.3 with $\varphi(t) = \frac{t}{4}$. Thus all the conditions of the Theorem 4.3 are satisfy. Then f and S have infinitely coincidence point in X. Indeed, $C(f, S) = \left[1, \frac{1+\sqrt{9}}{2}\right]$. Also, we can see that f and T are weakly compatible at a point a and $ffa = fa$ for $a = 1 \in C(f, T)$. Therefore all the conditions of theorem are satisfied. Therefore, a point 1 is a unique common fixed point of f and T in X.

***Corollary 4.5:-*** Let (X, d, s) be a b-metric space. Let $f : X \to X$ be a single-valued mapping and $S : X \to B(X)$ be a multi-valued mapping satisfying the following conditions:

1. the pair (S, f) satisfy the ($CLR_f$)-property,
2. for all $x, y \in X$,

$$H(Sx, Sy) \leq \varphi\Big(\max\Big\{d(fx, fy), d(fx, Sx), d(fy, Sy), \frac{1}{2s}[d(fx, Sy) + d(fy, Sx)], \frac{d(fx,Sx)(1+d(fy,Sy))}{1+d(fx,fy)}, \frac{d(fx,Sy)(1+d(fy,Sx))}{1+d(fx,fy)}\Big\}\Big).$$

If f(X) is subsets of X, then we have the following:

1. f and S have a coincidence point,
2. f and S have a common fixed point provided that f and S are weakly compatible at v and $ffv = fv$ for any $v \in C(f, S)$.

***Corollary 4.6:-*** Let (X, d, s) be a b-metric space. Let $f, g, S, T : X \to X$ be four single-valued mappings satisfying the following conditions:

1. the pair (S, f) and (T, g) satisfy the ($CLR_f$)-property,
2. for all $x, y \in X$,

$$d(Sx, Ty) \leq \varphi\Big(\max\Big\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2s}[d(fx, Ty) + d(gy, Sx)], \frac{d(fx,Sx)(1+d(gy,Ty))}{1+d(fx,gy)}, \frac{d(fx,Ty)(1+d(gy,Sx))}{1+d(fx,gy)}\Big\}\Big).$$

If f(X) and g(X) are subsets of X, then we have the following:

1. f and S have a coincidence point,
2. g and T have a coincidence point,
3. f and S have a common fixed point provided that f and S are weakly compatible at v and $ffv = fv$ for any $v \in C(f, S)$,
4. g and T have a common fixed point provided that g and T are weakly compatible at v and $ggv = gv$ for any $v \in C(g, T)$,
5. f, g, S, T have a common point provided that both (3) and (4) are true.

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