GENERALIZED INVEXITY AND INVARIANT MONOTONICITY

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ABSTRACT

In this paper, several kinds of invariant monotone maps and generalized invariant monotone maps are introduced. Some examples are given which show that invariant monotonicity and generalized invariant monotonicity are proper generalizations of monotonicity and generalized monotonicity. Relationships between generalized invariant monotonicity and generalized invexity are established. Our results are generalizations of those presented by Pradhan S.K., and D.K. Dalai.

INTRODUCTION

Convexity is a common assumption made in Mathematical programming. In recent years, there have been increasing attempts to weaken the convexity conditions. Consequently, several classes of (generalized) invex functions have been introduced in the literature. More specifically, the concept of invexity was introduced by Hanson [2], where it is shown that the Kuhn-Tucker conditions are sufficient for (global) optimality of non-linear programming problems under invexity condition. Weir and Mond [5]-[6] introduced the concept of pre invex functions and applied it to the establishment of the sufficient optimality conditions and duality in (multi objective) non-linear programming. Mohan and Neogy [3] shown that, under certain conditions an invex function is preinvex.

In this paper we further generalized the idea of X.M. Yang, X.M. Yang and K.L. Teo [7] taking into account of four variables instead of three variables. We introduced several types of generalized invariant monotonicities which are generalization of the (strict) monotonicity. The main aim of this chapter is to establish relations among generalized invariant monotonicities and generalized invexities.

Invariant Monotone Maps And Strictly Invariant Monotone Maps

Let $\Gamma$ be non-empty subset of $\mathbb{R}^n$. Let $\eta$ be a vector valued function from $X \times X \times X \times X$ into $\mathbb{R}^n$ ($X \subseteq \mathbb{R}^n$) and Let $F$ be a vector function from $\Gamma$ into $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$.

**Definition:** A set $\Gamma$ is said to be invex with respect to $\eta$ if there exists an $\eta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for any $p, q, r, s \in \Gamma$ and $\lambda \in [0, 1]$,

$p + \lambda \eta (p, q, r, s) \in \Gamma$
$q + \lambda \eta (s, p, q, r) \in \Gamma$

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\[ r + \lambda \eta (q, r, s, p) \in \Gamma \]
\[ s + \lambda \eta (r, s, p, q) \in \Gamma \]

**Definition**

F is said to be Monotone on \( \Gamma \) if for every pair of points \( p, q, r, s \in \Gamma \),
\[ (q - p)^T (F(q) - F(p)) \geq 0. \]

**Definition**

Let \( \Gamma \) be an invex set with respect to \( \eta \). F is said to be invariant Monotone on \( \Gamma \) with respect to \( \eta \) if for any \( p, q, r, s \in \Gamma \),
\[ \eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T F(r) + \eta(r, s, p, q)^T F(q) + \eta(q, r, s, p)^T F(p) \leq 0. \]

**Remark**

Every Monotone map is an invariant Monotone Map with
\[ \eta(p, q, r, s) = 2s - p - r, \]
\[ \eta(s, p, q, r) = 2r - q - s, \]
\[ \eta(r, s, p, q) = 2q - r - p, \]
\[ \eta(q, r, s, p) = 2p - q - s. \]

F is monotone on \( \Gamma \), we have for every pair of points \( p, q, r, s \in \Gamma \),
\[ (s - r)^T F(s) + (r - q)^T F(r) + (q - p)^T F(q) + (p - s)^T F(p) \geq 0 \]
\[ \Rightarrow [F(s) - F(r)]^T (s - r) + [F(r) - F(q)]^T (r - q) + [F(q) - F(p)]^T (q - p) + [F(p) - F(s)]^T (p - s) \leq 0 \]
\[ \Rightarrow (2s - r)^T F(s) + (2r - q)^T F(r) + (2q - p)^T F(q) + (2p - s)^T F(p) \leq 0 \]
\[ \Rightarrow \eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T F(r) + \eta(r, s, p, q)^T F(q) + \eta(q, r, s, p)^T F(p) \leq 0 \]

F is invariant Monotone with respect to \( \eta \) on \( \Gamma \), but the converse is not true.

**Example:** Let F and \( \eta \) be Maps defined as
\[ F(p) = (1 + \cos p_1, 1 + \cos p_2, 1 + \cos p_3, 1 + \cos p_4) \]
\[ F(q) = (1 + \cos q_1, 1 + \cos q_2, 1 + \cos q_3, 1 + \cos q_4) \]
\[ F(r) = (1 + \cos r_1, 1 + \cos r_2, 1 + \cos r_3, 1 + \cos r_4) \]
\[ F(s) = (1 + \cos s_1, 1 + \cos s_2, 1 + \cos s_3, 1 + \cos s_4) \]
\[ \eta(p, q, r, s) = \left( (1 + \cos q_1) \sin p_1 \sin q_1 \cos q_1, (1 + \cos q_2) \sin p_2 \sin q_2 \cos q_2, (1 + \cos q_3) \sin p_3 \sin q_3 \cos q_3, (1 + \cos q_4) \sin p_4 \sin q_4 \cos q_4 \right) \]
\[ \eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T F(r) + \eta(r, s, p, q)^T F(q) + \eta(q, r, s, p)^T F(p) \]
\[ = \left( (1 + \cos r_1) \sin p_1 \sin q_1 \cos r_1, (1 + \cos r_2) \sin p_2 \sin q_2 \cos r_2, (1 + \cos r_3) \sin p_3 \sin q_3 \cos r_3, (1 + \cos r_4) \sin p_4 \sin q_4 \cos r_4 \right) \]
\[ + \left( (1 + \cos q_1) \sin r_1 \sin p_1 \cos q_1, (1 + \cos q_2) \sin r_2 \sin p_2 \cos q_2, (1 + \cos q_3) \sin r_3 \sin p_3 \cos q_3, (1 + \cos q_4) \sin r_4 \sin p_4 \cos q_4 \right) \]
\[ + \left( (1 + \cos q_1) \sin r_2 \sin p_2 \cos q_2, (1 + \cos q_2) \sin r_3 \sin p_3 \cos q_3, (1 + \cos q_3) \sin r_4 \sin p_4 \cos q_4 \right) \]
\[ + \left( (1 + \cos q_1) \sin r_3 \sin p_3 \cos q_3, (1 + \cos q_2) \sin r_4 \sin p_4 \cos q_4 \right) \]
\[ + \left\{ \begin{array}{l} (1 + \cos p_1)(\sin q_1 - \sin r_1) \sin p_1 + (1 + \cos p_2)(\sin q_2 - \sin r_2) \sin p_2 \\ (1 + \cos p_3)(\sin q_3 - \sin r_3) \sin p_3 + (1 + \cos p_4)(\sin q_4 - \sin r_4) \sin p_4 \end{array} \right\} \]

\[ = \sum \left\{ \frac{\sin p_1 \sin r_1 - \sin q_1 \sin r_1}{\cos r_1} + \frac{\sin r_1 \sin q_1 - \sin p_1 \sin q_1}{\cos q_1} + \frac{\sin q_1 \sin q_1 - \sin p_1 \sin q_1}{\cos p_1} \right\} \]

\[ = \sum \left\{ \sin p_1 \sin r_1 \cos q_1 \cos r_1 + \sin r_1 \sin q_1 \cos p_1 \cos q_1 + \sin q_1 \sin q_1 \cos p_1 \cos q_1 \right\} \]

\[ = \sum \left\{ \tan p_1 \tan r_1 \cos p_1 \cos r_1 + \tan q_1 \tan r_1 \cos q_1 \cos r_1 + \tan p_1 \tan q_1 \cos p_1 \cos q_1 \right\} \]

\[ = \sum \left\{ -2 \left( \tan p_1 \tan r_1 \sin \left( \frac{p_1 + r_1}{2} \right) \sin \left( \frac{p_1 - r_1}{2} \right) + \tan q_1 \tan r_1 \sin \left( \frac{q_1 + r_1}{2} \right) \sin \left( \frac{q_1 - r_1}{2} \right) \right) \right\} \]

\[ \text{for } p, q, r, s \in \left( 0, \frac{\pi}{2} \right) \times \left( 0, \frac{\pi}{2} \right) \times \left( 0, \frac{\pi}{2} \right) \times \left( 0, \frac{\pi}{2} \right) \]

Hence \( F \) is invariant monotone with respect to \( \eta \).

Let \( p = \left( \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4} \right) \), \( q = \left( \frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6} \right) \)

Now, \( (q - p)^T [F(q) - F(p)] \)

\[ = \left( \frac{\pi}{6} \right) \left( \frac{\pi}{4} \right) \left( \left[ 1 + \cos \frac{\pi}{6}, 1 + \cos \frac{\pi}{6}, 1 + \cos \frac{\pi}{6}, 1 + \cos \frac{\pi}{6} \right] - \left[ 1 + \cos \frac{\pi}{4}, 1 + \cos \frac{\pi}{4}, 1 + \cos \frac{\pi}{4}, 1 + \cos \frac{\pi}{4} \right] \right) \]

\[ = \left( \frac{\pi}{12} \right) \sqrt{4 \left( \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \right)^2} \]

\[ = \left( \frac{\pi}{6} \right) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) < 0 \]

Thus \( F \) is not Monotone.

**Definition**

Let the set \( \Gamma \) be invex with respect to \( \eta \) and \( \eta : X \times X \times X \times X \rightarrow \mathbb{R}^4 \) be a vector valued function. The function \( f : \Gamma \rightarrow \mathbb{R} \) is said to be pre-invex in respect to \( \eta \) if

\[
\begin{align*}
& f(p + \lambda \eta(p, q, r, s)) \leq \lambda f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s) \\
& f(q + \lambda \eta(s, p, q, r)) \leq \lambda f(q) + \lambda_1 f(p) + \lambda_2 f(q) + \lambda_3 f(r) \\
& f(r + \lambda \eta(q, r, s, p)) \leq \lambda f(q) + \lambda_1 f(r) + \lambda_2 f(s) + \lambda_3 f(p) \\
& f(s + \eta \lambda(r, s, p, q)) \leq \lambda f(s) + \lambda_1 f(s) + \lambda_2 f(p) + \lambda_3 f(q) \\
& \forall \ p, q, r, s \in \Gamma \text{ where } \lambda + \lambda_1 + \lambda_2 + \lambda_3 = 1; \lambda \in [0, 1].
\end{align*}
\]
**Assumption - A**

Let the set $\Gamma$ be convex with respect to $\eta$ and let $f : \Gamma \rightarrow \mathbb{R}$. Then

$$f(s + \lambda \eta(p, q, r, s)) \leq f(p)$$

$$f(r + \lambda \eta(s, p, q, r)) \leq f(q)$$

$$f(q + \lambda \eta(q, r, s, p)) \leq f(r)$$

$$f(p + \lambda \eta(r, s, p, q)) \leq f(s)$$

**Remark**

Assumption A is just the inequality of the definition of pre-invexity with $\lambda = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$

**Assumption - B**

Let $\eta : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}^n$ and then for any $p, q, r, s \in \mathbb{R}^n$ and for $\lambda \in [0, 1]$, $\eta(q, q, q + \lambda \eta(p, q, r, s)) = \lambda \eta(p, q, r, s)$

$$\eta(p, q + \lambda_1 \eta, r, s) = (1 - \lambda_1) \eta(p, q, r, s)$$

$$\eta(p, q, r + \lambda_2 \eta, s) = (1 - \lambda_2) \eta(p, q, r, s)$$

$$\eta(p, q, r, s + \lambda_3 \eta) = (1 - \lambda_3) \eta(p, q, r, s),$$

where $\lambda + \lambda_1 + \lambda_2 + \lambda_3 = 1$.

**Remark:** We will show that Assumption B holds if

$$\eta(p, q, r, s) = (3p - q - r - s) + o(3p - q - r - s)$$

\begin{enumerate}
\item \(\eta(q, q, q + \lambda \eta(p, q, r, s)) = \lambda \eta(p, q, r, s)\)
\item \(\eta(p, q + \lambda_1 \eta, r, s) = (3p - q - r - s) + o(3p - q - r - s)\)
\item \(\eta(p, q, r + \lambda_2 \eta, s) = (3p - q - r - s) + o(3p - q - r - s)\)
\item \(\eta(p, q, r, s + \lambda_3 \eta) = (3p - q - r - s) + o(3p - q - r - s)\)
\end{enumerate}
Thus, f is invex with respect to \( \eta \) on \( K \) and that \( f \) and \( \eta \) satisfy Assumptions A and B. However, \( f \) is not convex.

The following theorem shows that the preinvexity of a function is equivalent to the invariant monotone property of its gradient. This is a generalization of the convexity of a function and the monotonicity of its gradient obtained in [1].

**Lemma**

Let \( f \) be differentiable on an open set containing \( \Gamma \). If \( f \) is invex with respect to \( \eta \), then \( \nabla f \) is an invariant monotone with respect to \( \eta \).

**Proof**

Let \( f \) be invex. Then we have

\[
\frac{f(p + \lambda \eta(p, q, r, s)) - f(p)}{\lambda} \leq \lambda f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s),
\]

where \( p, q, r, s \in \Gamma, \lambda + \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda \in [0, 1] \).

Applying Mean Value Theorem, \( f(p + \lambda \eta) - f(p) = \lambda \eta f'(p + \theta \lambda \eta) \)

\[
\Rightarrow f'(p + \lambda \eta) - f(p) = \frac{\lambda \eta}{\lambda} = f'(p + \theta \lambda \eta) \tag{1}
\]

Here, \( f(p + \lambda \eta(p, q, r, s)) - f(p) \leq \frac{\lambda}{\lambda_1} f(p) + \frac{\lambda_1}{\lambda_2} f(q) + \frac{\lambda_2}{\lambda_3} f(r) + \frac{\lambda_3}{\lambda_3} f(s) \)

\[
\Rightarrow \frac{f(p + \lambda \eta(p, q, r, s)) - f(p)}{\lambda \eta} \leq \frac{(\lambda - 1) f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)}{\lambda \eta}
\]

From equation (1), we get

\[
f'(p + \theta \lambda \eta) \leq \frac{(\lambda - 1) f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)}{\lambda \eta}
\]

Adding (i), (ii), (iii) and (iv), we have

\[
\eta(p, q, r, s)^T \nabla f(s) + \eta(s, p, q, r)^T \nabla f(r) + \eta(q, r, s, p)^T \nabla f(q) + \eta(r, s, p, q)^T \nabla f(p)
\]
\[ (\lambda - 1 + \lambda_3 + \lambda_2 + \lambda_1) f(p) + (\lambda_2 - 1 + \lambda_1 + \lambda + \lambda_3) f(q) + (\lambda_3 - 1 + \lambda_2 + \lambda + \lambda_1) f(r) \]

\[ + (\lambda - 1 + \lambda_3 + \lambda_2 + \lambda_1) f(s) \leq 0, \text{ for } \lambda + \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda \in [0, 1] \]

\( F \) is invariant monotone on \( \Gamma \) with respect to \( \eta \).

**Theorem**

Let \( f \) and \( \eta \) satisfy respectively Assumptions A and B and let \( f \) be differentiable on \( \Gamma \). Then \( f \) is a preinvex function with respect to \( \eta \) on \( \Gamma \) if and only if \( F \) is invariant monotone with respect to \( \eta \) on \( \Gamma \) and \( f \) satisfies Assumption A.

**Proof**

Suppose \( f \) is preinvex on \( \Gamma \) with respect to \( \Gamma \). Then

\[ f(p + \lambda \eta(p, q, r, s)) \leq \lambda f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s) \]

Applying Mean Value Theorem.

\[ f(p + \lambda \eta(p, q, r, s)) - f(p) = \lambda \eta f'((p + \theta \lambda \eta)) \]

where \( 0 < \theta < 1 \)

Now,

\[ f(p + \lambda \eta(p, q, r, s)) - f(p) > \lambda f(p) - f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s) \]

\[ > (\lambda - 1) f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s) \]

\[ \Rightarrow f(p + \lambda \eta(p, q, r, s)) - f(p) > \frac{(\lambda - 1) f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)}{\lambda \eta} \]

Using equation (1), we get

\[ f'(p + \theta \lambda \eta) > \frac{(\lambda - 1) f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)}{\lambda \eta} \]

\[ \Rightarrow \lambda \eta^T \nabla f(p) > (\lambda - 1) f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s) \] (i)

Similarly, we get

\[ \lambda \eta^T \nabla f(q) > (\lambda_2 - 1) f(q) + \lambda_2 f(s) + \lambda_1 f(p) + \lambda_3 f(r) \] (ii)

\[ \lambda \eta^T \nabla f(r) > (\lambda_1 - 1) f(r) + \lambda_2 f(q) + \lambda_3 f(s) + \lambda_3 f(p) \] (iii)

\[ \lambda \eta^T \nabla f(s) > (\lambda_1 - 1) f(s) + \lambda_2 f(r) + \lambda_2 f(p) + \lambda_3 f(q) \] (iv)

Adding equation (i), (ii), (iii) and (iv) we have

\[ \eta(p, q, r, s)^T \nabla f(p) + \eta(s, p, q, r)^T \nabla f(r) + \eta(q, r, s, p)^T \nabla f(q) + \eta(r, s, p, q)^T \nabla f(p) > \]

\[ (\lambda - 1 + \lambda_1 + \lambda_2 + \lambda_3) f(p) + (\lambda_2 - 1 + \lambda_1 + \lambda + \lambda_3) f(q) + (\lambda_3 - 1 + \lambda_2 + \lambda + \lambda_1) f(r) + (\lambda_1 - 1 + \lambda_2 + \lambda + \lambda_3) f(s) = 0, \]

which contradicts the invariant monotonicity with respect to \( \eta \) on \( \Gamma \).

**Definition**

\( F \) is said to be strictly monotone on \( \Gamma \) if for every pair of distinct point \( (p, q) \in \Gamma \),

\[ (q - p)^T (F(q) - F(p)) > 0 \]
Definition

Let $\Gamma$ be an invex set with respect to $\eta$. $F$ is said to be strictly invariant monotone with respect to $\eta$ on $\Gamma$ if for any point $p, q, r, s \in \Gamma$,
\[
\eta(p, q, r, s)^T \triangledown f(s) + \eta(s, p, q, r)^T \triangledown f(r) + \eta(q, r, s, p)^T \triangledown f(p) + \eta(r, s, p, q)^T \triangledown f(q) \leq 0
\]

Remark

Every monotone map is an invariant monotone map with
\[
\eta(p, q, r, s) = 2s - p - r,
\eta(q, r, s, p) = 2r - q - s
\eta(s, p, q, r) = 2p - q - s,
\eta(r, s, p, q) = 2q - p - r.
\]
$F$ is monotone on $\Gamma$.

We have for every pair of points $p, q, r, s \in \Gamma$,
\[
(s - r)^T F(s) + (r - q)^T F(r) + (q - p)^T F(q) + (p - s)^T F(p) > 0
\]
\[
\Rightarrow (F(s) - F(r))(s - r) + (F(r) - F(q))(r - q) + (F(q) - F(p))(q - p) + (F(p) - F(s))(p - s) \leq 0
\]
\[
\Rightarrow (2s - p - r) F(s) + (2r - s - q) F(r) + (2q - r - p) F(q) + (2p - q - s) F(p) \leq 0
\]
\[
\Rightarrow \eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T F(r) + \eta(q, r, s, p)^T F(q) + \eta(r, s, p, q)^T F(p) \leq 0
\]
$F$ is invariant monotone with respect to $\eta$ on $\Gamma$ but the converse is not true.

Example

Define the map $\Gamma$ and $\eta$ as
\[
F(p) = (-1 - \cos p_1, -1 - \cos p_2, -1 - \cos p_3, -1 - \cos p_4)
\]
\[
F(q) = (-1 - \cos q_1, -1 - \cos q_2, -1 - \cos q_3, -1 - \cos q_4)
\]
\[
F(r) = (-1 - \cos r_1, -1 - \cos r_2, -1 - \cos r_3, -1 - \cos r_4)
\]
\[
F(s) = (-1 - \cos s_1, -1 - \cos s_2, -1 - \cos s_3, -1 - \cos s_4)
\]
\[
\eta(p, q, r, s) = \left[(-1 - \cos q_1)(\sin p_1 - \sin q_1) \cdot \sin s_1 \cos q_1, (-1 - \cos q_2)(\sin p_2 - \sin q_2) \cdot \sin s_2 \cos q_2, \right.
\]
\[
(-1 - \cos q_3)(\sin p_3 - \sin q_3) \cdot \sin s_3 \cos q_3, (-1 - \cos q_4)(\sin p_4 - \sin q_4) \cdot \sin s_4 \cos q_4\right]
\]
where $p, q, r, s \in \left\{0, \frac{\pi}{2} \right\} \times \left\{0, \frac{\pi}{2} \right\} \times \left\{0, \frac{\pi}{2} \right\} \times \left\{0, \frac{\pi}{2} \right\}$
\[
\eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T F(r) + \eta(q, r, s, p)^T F(q) + \eta(r, s, p, q)^T F(p)
\]
\[
= \left[(-1 - \sin s_1) \left(\frac{\cos q_1 - \cos p_1}{\sin s_1}\right) + (-1 - \sin s_2) \left(\frac{\cos q_2 - \cos p_2}{\sin s_2}\right) \right]
\]
\[
+ (-1 - \sin s_3) \left(\frac{\cos q_3 - \cos p_3}{\sin s_3}\right) + (-1 - \sin s_4) \left(\frac{\cos q_4 - \cos p_4}{\sin s_4}\right) \right]
\]
\[
+ \left[(-1 - \sin q_1) \left(\frac{\cos p_1 - \cos s_1}{\sin q_1}\right) + (-1 - \sin q_2) \left(\frac{\cos p_2 - \cos s_2}{\sin q_2}\right) \right]
\]
\[
+ (-1 - \sin q_3) \left(\frac{\cos p_3 - \cos s_3}{\sin q_3}\right) + (-1 - \sin q_4) \left(\frac{\cos p_4 - \cos s_4}{\sin q_4}\right) \right]
\]
\[
+ \left[(-1 - \sin p_1) \left(\frac{\cos s_1 - \cos q_1}{\sin p_1}\right) + (-1 - \sin p_2) \left(\frac{\cos s_2 - \cos q_2}{\sin p_2}\right) \right]
\]
\[
+ (-1 - \sin p_3) \left(\frac{\cos s_3 - \cos q_3}{\sin p_3}\right) + (-1 - \sin p_4) \left(\frac{\cos s_4 - \cos q_4}{\sin p_4}\right) \right]
\]
+ \left[ (-1 - \sin r_1) \left( \frac{\cos s_1 - \cos p_1}{\sin r_1} \right) + (-1 - \sin r_2) \left( \frac{\cos s_2 - \cos p_2}{\sin r_2} \right) \right]

+ (-1 - \sin r_3) \left( \frac{\cos s_3 - \cos p_3}{\sin r_3} \right) + (-1 - \sin r_4) \left( \frac{\cos s_4 - \cos p_4}{\sin r_4} \right)

= \sum (-1 - \sin s_j) \left( \frac{\cos q_j - \cos p_1}{\sin s_j} \right) + (-1 - \sin q_i) \left( \frac{\cos p_1 - \cos s_i}{\sin q_i} \right)

+ (-1 - \sin p_i) \left( \frac{\cos s_i - \cos q_i}{\sin p_i} \right) + (-1 - \sin r_j) \left( \frac{\cos s_i - \cos r_i}{\sin r_i} \right)

= \sum \left[ \frac{2}{\sin s_i} \left( \sin \left( \frac{p_1 + q_1}{2} \right) \sin \left( \frac{s_i - p_1}{2} \right) \right) \right]

+ \left[ \frac{2}{\sin q_i} \left( \sin \left( \frac{q_i + s_i}{2} \right) \sin \left( \frac{s_i - q_i}{2} \right) \right) \right]

+ \left[ \frac{2}{\sin p_i} \left( \sin \left( \frac{p_i + s_i}{2} \right) \sin \left( \frac{s_i - p_i}{2} \right) \right) \right]

+ \left[ \frac{2}{\sin r_i} \left( \sin \left( \frac{r_i + s_i}{2} \right) \sin \left( \frac{s_i - r_i}{2} \right) \right) \right]

\text{Clearly } F \text{ is strictly invariant monotone with respect to } \eta.

\text{Let } p = \left( \frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6} \right), \ q = \left( \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4} \right)

\text{Then} \ (q - p)^T (F(q) - F(p))

= \left[ \frac{\pi}{4} - \frac{\pi}{6} \right] \left[ \begin{array}{c}
-1 - \sin \frac{\pi}{4},
-1 - \sin \frac{\pi}{4},
-1 - \sin \frac{\pi}{4},
-1 - \sin \frac{\pi}{4},
-1 - \sin \frac{\pi}{4},
-1 - \sin \frac{\pi}{4},
-1 - \sin \frac{\pi}{4},
-1 - \sin \frac{\pi}{4},
\end{array} \right]

= \pi \left[ \frac{1}{6} - \frac{1}{\sqrt{2}} \right]

Thus \ F \text{ is not strictly monotone.}

\textbf{Remark}

Every strictly invariant monotone map is an invariant monotone map with respect to the same \( \eta \) but the converse is not necessarily true.

\textbf{Example}

Define the map \( F \) and \( \eta \) as

\( F = \left[ \begin{array}{c}
\cos \left( \frac{\pi}{4} \right),
\cos \left( \frac{\pi}{4} \right),
\cos \left( \frac{\pi}{4} \right),
\cos \left( \frac{\pi}{4} \right),
\cos \left( \frac{\pi}{4} \right),
\cos \left( \frac{\pi}{4} \right),
\cos \left( \frac{\pi}{4} \right),
\cos \left( \frac{\pi}{4} \right),
\end{array} \right] \)

\( \eta = \left[ \begin{array}{c}
\frac{\pi}{6},
\frac{\pi}{6},
\frac{\pi}{6},
\frac{\pi}{6},
\frac{\pi}{6},
\frac{\pi}{6},
\frac{\pi}{6},
\frac{\pi}{6},
\end{array} \right] \)
F(p) = (− cos p₁, − cos p₂, − cos p₃, − cos p₄)

Where

p₁, p₂, p₃, p₄ ∈ \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]

η(p, q, r, s) = \left[\frac{\sin p₁ - \sin q₁ - \sin r₁}{\cos s₁}, \frac{\sin p₂ - \sin q₂ - \sin r₂}{\cos s₂}, \frac{\sin p₃ - \sin q₃ - \sin r₃}{\cos s₃}, \frac{\sin p₄ - \sin q₄ - \sin r₄}{\cos s₄}\right]

p₁, q₁, r₁, s₁ ∈ \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]

\eta(p, q, r, s)ᵀ F(s) + η(q, r, s, p)ᵀ F(q) + η(r, s, p, q)ᵀ F(p)

= \frac{\sin p₁ - \sin q₁ - \sin r₁}{\cos s₁} (-\cos s₁) + \frac{\sin p₂ - \sin q₂ - \sin r₂}{\cos s₂} (-\cos s₂)

+ \frac{\sin p₃ - \sin q₃ - \sin r₃}{\cos s₃} (-\cos s₃) + \frac{\sin p₄ - \sin q₄ - \sin r₄}{\cos s₄} (-\cos s₄)

+ \frac{\sin q₁ - \sin r₁ - \sin s₁}{\cos p₁} (-\cos p₁) + \frac{\sin q₂ - \sin r₂ - \sin s₂}{\cos p₂} (-\cos p₂)

+ \frac{\sin q₃ - \sin r₃ - \sin s₃}{\cos p₃} (-\cos p₃) + \frac{\sin q₄ - \sin r₄ - \sin s₄}{\cos p₄} (-\cos p₄)

+ \frac{\sin r₁ - \sin s₁ - \sin p₁}{\cos q₁} (-\cos q₁) + \frac{\sin r₂ - \sin s₂ - \sin p₂}{\cos q₂} (-\cos q₂)

+ \frac{\sin r₃ - \sin s₃ - \sin p₃}{\cos q₃} (-\cos q₃) + \frac{\sin r₄ - \sin s₄ - \sin p₄}{\cos q₄} (-\cos q₄)

+ \frac{\sin s₁ - \sin p₁ - \sin q₁}{\cos r₁} (-\cos r₁) + \frac{\sin s₂ - \sin p₂ - \sin q₂}{\cos r₂} (-\cos r₂)

+ \frac{\sin s₃ - \sin p₃ - \sin q₃}{\cos r₃} (-\cos r₃) + \frac{\sin s₄ - \sin p₄ - \sin q₄}{\cos r₄} (-\cos r₄) = 0

Clearly F is invariant monotone with respect to η.
Thus F is not strictly invariant monotone with respect to η.

**Theorem**

Let f and η satisfy Assumptions A and B respectively. Then f is a strictly preinvex function with respect to η on Γ if and only if ∀f is strictly invariant monotone with respect to η on Γ and f satisfies Assumption A.

**Proof**

Suppose f is strictly preinvex on Γ with respect to η, then

f(p + λ₁f(p, q, r, s)) < λ₁f(p) + λ₂F(r) + λ₃(f(q),)

f(q + λ₂f(s, p, q, r)) < λ₂f(s) + λ₁f(p) + λ₃f(q) + λ₃f(r),

f(r + λ₃f(q, r, s, p)) < λ₃f(q) + λ₁f(r) + λ₂f(s) + λ₃f(p),

f(s) + λ₃f(r, s, p, q) < λ₃f(r) + λ₁f(s) + λ₂f(p) + λ₃f(q).

First assumption A is just inequality with λ₁ = 1, λ₂ = λ₃ = 0 in the above inequalities and lemma 2.1, it follows that f is strictly invariant monotone with respect to η on Γ.

Conversely, suppose that ∀f is strictly invariant monotone with respect to η on Γ. Assume that f is not strictly preinvex with respect to η on Γ. Then there exist. p, q, r, s ∈ Γ such that

f(p + η(p, q, r, s)) ≥ λ₁f(p) + λ₂f(r) + λ₃f(s),

where p, q, r, s ∈ Γ such that λ₁ + λ₂ + λ₃ = 1 and λ ∈ [0, 1]

Applying Mean Value Theorem,

f(p + η) - f(p) = λ η f'(p + θλη), when 0 < θ < 1
\[ f(p + \lambda \eta) - f(p) \leq \lambda \eta \]
\[ \frac{f(p + \lambda \eta(p, q, r, s) - f(p)}{\lambda \eta} \geq \frac{(\lambda - 1)f(p) + \lambda f(q) + \lambda f(r) + \lambda f(s)}{\lambda \eta} \]

By using Equation \( (1) \), we get
\[ f'(p + \theta \lambda \eta) \geq \frac{(\lambda - 1)f(p) + \lambda f(q) + \lambda f(r) + \lambda f(s)}{\lambda \eta} \]

which contradicts the strictly invariant monotonicity with respect to \( \eta \) on \( \nabla f \).

References

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