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ZEROS OF POLYNOMIALS

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ABSTRACT

In this paper we find regions containing all or a specific number of zeros of a polynomial in which the real and imaginary parts of the coefficients satisfy some restricted conditions.

INTRODUCTION

In connection with the famous Enestrom-Kakeya Theorem [9,10] which states that all the zeros of a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ with $a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0$ lie in $|z| \le 1$, the following results were recently proved by Gulzar et al [6,7] :

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\sim_j,\ \operatorname{Im}(a_j)=\ \ _j$,

$j=0,1,2,......,n$ such that for some $ª, «; 0 \le ª \le n-1, 0 \le « \le n-1$ and for some $k_1, k_2 \le 1; ²_1, ²_2 \ge 1$,

$k_1 \sim_n \le \sim_{n-1} \le ...... \le ²_1 \sim_ª$

$k_2 \ \ _n \le \ \ _{n-1} \le ...... \le ²_2 \ \ _«$,

and

$L = |\sim_ª - \sim_{ª-1}| + |\sim_{ª-1} - \sim_{ª-2}| + ... + |\sim_1 - \sim_0| + |\sim_0|$,

$M = |\ \ _« - \ \ _{«-1}| + |\ \ _{«-1} - \ \ _{«-2}| + ... + |\ \ _1 - \ \ _0| + |\ \ _0|$,

Then all the zeros of P(z) lie in

$$\left| z - \frac{(1-k_1)\sim_n + i(1-k_2)\ \ _n}{a_n} \right| \le \frac{1}{|a_n|}[²_1(\sim_ª + |\sim_ª|) + ²_2(\ \ _« + |\ \ _«|) - |\sim_ª| - |\ \ _«| - k_1\sim_n - k_2\ \ _n + L + M].$$

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\sim_j,\ \operatorname{Im}(a_j)=\ \ _j$,

$j=0,1,2,......,n$ such that for some $ª, «; 0 \le ª \le n-1, 0 \le « \le n-1$ and for some $k_1, k_2 \le 1; ²_1, ²_2 \ge 1$,

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$k_1 \sim_n \le \sim_{n-1} \le ...... \le {}^{2}{}_1 \sim_{ª}$

$k_2 \quad_n \le \quad_{n-1} \le ...... \le {}^{2}{}_2 \quad_{«}$,

and

$L = |\sim_{ª} - \sim_{ª-1}| + |\sim_{ª-1} - \sim_{ª-2}| + ... + |\sim_1 - \sim_0| + |\sim_0|$,

$M = |\quad_{«} - \quad_{«-1}| + |\quad_{«-1} - \quad_{«-2}| + ... + |\quad_1 - \quad_0| + |\quad_0|$,

Then the number of zero of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c} \log \frac{A}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c} \log \frac{B}{|a_0|}$ for $R \le 1$, where

$X = |a_n| R^{n+1} + R^n [|\sim_n| + |\quad_n| - k_{1(}(|\sim_n| + \sim_n) - k_2(|\quad_n| + \quad_n) + {}^{2}{}_1(|\sim_{ª}| + \sim_{ª})$

$+ {}^{2}{}_2(|\quad_{«}| + \quad_{«}) - |\sim_{ª}| - |\quad_{«}| + L + M - |\sim_0| - |\quad_0|\}$,

$Y = |a_n| R^{n+1} + R[|\sim_n| + |\quad_n| - k_{1(}(|\sim_n| + \sim_n) - k_2(|\quad_n| + \quad_n) + {}^{2}{}_1(|\sim_{ª}| + \sim_{ª})$

$+ {}^{2}{}_2(|\quad_{«}| + \quad_{«}) - |\sim_{ª}| - |\quad_{«}| + L + M] - (1 - R)(|\sim_0| + |\quad_0|)$,

$A = |a_n| R^{n+1} + R^n [|\sim_n| + |\quad_n| - k_{1(}(|\sim_n| + \sim_n) - k_2(|\quad_n| + \quad_n) + {}^{2}{}_1(|\sim_{ª}| + \sim_{ª})$

$+ {}^{2}{}_2(|\quad_{«}| + \quad_{«}) - |\sim_{ª}| - |\quad_{«}| + L + M]$,

$B = |a_n| R^{n+1} + R[|\sim_n| + |\quad_n| - k_{1(}(|\sim_n| + \sim_n) - k_2(|\quad_n| + \quad_n) + {}^{2}{}_1(|\sim_{ª}| + \sim_{ª})$

$+ {}^{2}{}_2(|\quad_{«}| + \quad_{«}) - |\sim_{ª}| - |\quad_{«}| + L + M] - (1 - R)(|\sim_0| + |\quad_0|)$,

R being any positive number.

MAIN RESULTS

In this paper we prove the following results:

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $ª, 0 \le ª \le n-1$ and for some $k \le 1; {}^{2} \ge 1$,

$k|a_n| \le |a_{n-1}| \le ...... \le |a_{ª+1}| \le {}^{2}|a_{ª}|$

and for some real $\sim$, ,

$|\arg a_j - \quad| \le \sim \le \frac{®}{2}, j = ª, ª + 1, ..., n$.

Then all the zeros of P(z) lie in

$|z - (1-k)| \le \frac{1}{|a_n|}[k|a_n|(\sin\sim - \cos\sim) + {}^{2}|\sim_{ª}|(\cos\sim + \sin\sim + 1)) - |\sim_{ª}| + L + 2\sin\sim \sum_{j=ª+1}^{n-1} |a_j|]$,

where

$L = |a_{ª} - a_{ª-1}| + |a_{ª-1} - a_{ª-2}| + ... + |a_1 - a_0| + |a_0|$.

Theorem 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $ª, 0 \le ª \le n-1$ and for some $k \le 1; {}^{2} \ge 1$,

$k|a_n| \le |a_{n-1}| \le ...... \le |a_{ª+1}| \le {}^{2}|a_{ª}|$

and for some real $\sim$, ,

$|\arg a_j - \quad| \le \sim \le \frac{®}{2}, j = ª, ª + 1, ..., n$.

Then the number of zeros of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{A}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{B}{|a_0|}$ for $R \le 1$, where

$$X = |a_n|R^{n+1} + R^n[|a_n| - k|a_n|(\cos\sim - \sin\sim + 1) - ²|\sim_ª|(\cos\sim - \sin\sim - 1) - |\sim_ª| + L$$
$$- |a_0| + 2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|],$$
$$Y = |a_n|R^{n+1} + R[|a_n| - k|a_n|(\cos\sim - \sin\sim + 1) - ²|\sim_ª|(\cos\sim - \sin\sim - 1) - |\sim_ª| + L$$
$$+ 2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|] - (1-R)|a_0|,$$
$$A = |a_n|R^{n+1} + R^n[|a_n| - k|a_n|(\cos\sim - \sin\sim + 1) - ²|\sim_ª|(\cos\sim - \sin\sim - 1) - |\sim_ª| + L$$
$$+ 2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|],$$
$$B = |a_n|R^{n+1} + R[|a_n| - k|a_n|(\cos\sim - \sin\sim + 1) - ²|\sim_ª|(\cos\sim - \sin\sim - 1) - |\sim_ª| + L$$
$$+ 2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|] - (1-R)|a_0|.$$

For different choices of the parameters in Theorems 1 and 2, we get many interesting results. For example takIng $² = 1$ in Theorem 1, we get the following result:

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $ª, 0 \le ª \le n-1$ and for some $k \le 1$,

$$k|a_n| \le |a_{n-1}| \le ...... \le |a_{ª+1}| \le |a_ª|$$

and for some real $\sim$, ,

$$|\arg a_j - \quad| \le \sim \le \frac{®}{2}, j = ª, ª+1, ..., n.$$

Then all the zeros of P(z) lie in

$$|z-(1-k)| \le \frac{1}{|a_n|}[k|a_n|(\sin\sim - \cos\sim) + |\sim_ª|(\cos\sim + \sin\sim) + L + 2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|],$$

where

$$L = |a_ª - a_{ª-1}| + |a_{ª-1} - a_{ª-2}| + ... + |a_1 - a_0| + |a_0|.$$

Taking $k = ² = 1$ in Theorem 1, we get the following result:

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $ª, 0 \le ª \le n-1,$,

$$|a_n| \le |a_{n-1}| \le ...... \le |a_{ª+1}| \le |a_ª|$$

and for some real $\sim$, ,

$$|\arg a_j - \quad| \le \sim \le \frac{®}{2}, j = ª, ª+1, ..., n.$$

Then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|a_n|}[k|a_n|(\sin\sim - \cos\sim) + |\sim_ª|(\cos\sim + \sin\sim) + L + 2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|],$$

where

$L=|a_\lambda-a_{\lambda-1}|+|a_{\lambda-1}-a_{\lambda-2}|+......+|a_1-a_0|+|a_0|$.

Taking $\lambda=0$ in Theorem 1, we get the following result:

Corollary 3: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $k\le 1;\rho\ge 1$,

$k|a_n|\le|a_{n-1}|\le......\le|a_{\lambda+1}|\le\rho|a_\lambda|$

and for some real α, β,

$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=\lambda,\lambda+1,......,n$.

Then all the zeros of P(z) lie in

$$|z-(1-k)|\le\frac{1}{|a_n|}[k|a_n|(\sin\alpha-\cos\alpha)+\rho|a_0|(\cos\alpha+\sin\alpha)+(1-\rho)|a_0|+2|a_0|+2\sin\alpha\sum_{j=\lambda+1}^{n-1}|a_j|],$$

Taking $\rho=1$ in Theorem 2, we get the following result:

Corollary 4: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $\lambda,0\le\lambda\le n-1$ and for some $k\le 1$,

$k|a_n|\le|a_{n-1}|\le......\le|a_{\lambda+1}|\le|a_\lambda|$

and for some real α, β,

$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=\lambda,\lambda+1,......,n$.

Then the number of zeros of P(z) in $\frac{|a_0|}{X}\le|z|\le\frac{R}{c},c>1$ is less than or equal to $\frac{1}{\log c}\log\frac{A}{|a_0|}$ for $R\ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{Y}\le|z|\le\frac{R}{c},c>1$ is less than or equal to $\frac{1}{\log c}\log\frac{B}{|a_0|}$ for $R\le 1$, where

$$X=|a_n|R^{n+1}+R^n[|a_n|-k|a_n|(\cos\alpha-\sin\alpha+1)-|\alpha_\lambda|(\cos\alpha-\sin\alpha)+L$$
$$-|a_0|+2\sin\alpha\sum_{j=\lambda+1}^{n-1}|a_j|],$$

$$Y=|a_n|R^{n+1}+R[|a_n|-k|a_n|(\cos\alpha-\sin\alpha+1)-|\alpha_\lambda|(\cos\alpha-\sin\alpha)+L$$
$$+2\sin\alpha\sum_{j=\lambda+1}^{n-1}|a_j|]-(1-R)|a_0|,$$

$$A=|a_n|R^{n+1}+R^n[|a_n|-k|a_n|(\cos\alpha-\sin\alpha+1)-|\alpha_\lambda|(\cos\alpha-\sin\alpha)+L$$
$$+2\sin\alpha\sum_{j=\lambda+1}^{n-1}|a_j|],$$

$$B=|a_n|R^{n+1}+R[|a_n|-k|a_n|(\cos\alpha-\sin\alpha+1)-|\alpha_\lambda|(\cos\alpha-\sin\alpha)+L$$
$$+2\sin\alpha\sum_{j=\lambda+1}^{n-1}|a_j|]-(1-R)|a_0|.$$

Taking $k=\rho=1$ in Theorem 2, we get the following result:

Corollary 5: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $\lambda,0\le\lambda\le n-1$,

$|a_n|\le|a_{n-1}|\le......\le|a_{\lambda+1}|\le|a_\lambda|$

and for some real $\sim$, ,

$$\left|\arg a_j - \quad\right| \le \sim \le \frac{®}{2},\ j = ª, ª+1, ..., n.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{A}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{B}{|a_0|}$ for $R \le 1$, where

$$X = |a_n|R^{n+1} + R^n[|a_n|(\sin\sim - \cos\sim) - |\sim_ª|(\cos\sim - \sin\sim) + L$$
$$-|a_0| + 2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|],$$
$$Y = |a_n|R^{n+1} + R[|a_n|(\sin\sim - \cos\sim) - |\sim_ª|(\cos\sim - \sin\sim) + L$$
$$+2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|] - (1-R)|a_0|,$$
$$A = |a_n|R^{n+1} + R^n[|a_n|(\sin\sim - \cos\sim) - |\sim_ª|(\cos\sim - \sin\sim) + L$$
$$+2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|],$$
$$B = |a_n|R^{n+1} + R[|a_n|(\sin\sim - \cos\sim) - |\sim_ª|(\cos\sim - \sin\sim) + L$$
$$+2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|] - (1-R)|a_0|.$$

Taking R=1 and $c = \frac{1}{¢}, 0 < ¢ < 1$ in Theorem 2, we get the following result:

Corollary 6: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $ª, 0 \le ª \le n-1$ and for some $k \le 1; ² \ge 1$,

$$k|a_n| \le |a_{n-1}| \le ...... \le |a_{ª+1}| \le ²|a_ª|$$

and for some real $\sim$, ,

$$\left|\arg a_j - \quad\right| \le \sim \le \frac{®}{2},\ j = ª, ª+1, ..., n.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{X} \le |z| \le ¢, 0 < ¢ < 1$ is less than or equal to $\frac{1}{\log\frac{1}{¢}}\log\frac{A}{|a_0|}$, where

$$X = |a_n| - |a_n|(\cos\sim - \sin\sim) - ²|\sim_ª|(\cos\sim - \sin\sim - 1) - |\sim_ª| + L$$
$$-|a_0| + 2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|,$$
$$A = |a_n| - |a_n|(\cos\sim - \sin\sim) - ²|\sim_ª|(\cos\sim - \sin\sim - 1) - |\sim_ª| + L \ + 2\sin\sim \sum_{j=ª+1}^{n-1}|a_j|].$$

Lemmas

For the proofs of the above results, we need the following lemmas:

Lemma 1: For any two complex numbers b_1, b_2 such that $|b_1| \ge |b_2|$ and $\left|\arg b_j - \quad\right| \le \sim \le \frac{®}{2},\ j = 1,2$

for some real $\sim$, ,

$$|b_1 - b_2| \le (|b_1| - |b_2|)\cos\alpha + (|b_1| + |b_2|)\sin\alpha .$$

The above lemma is due to Govil and Rahman [4].

Lemma 2: Let f(z) (not identically zero) be analytic for $|z| \le R,\ f(0) \ne 0$ and $f(a_k) = 0,\ \ k = 1,2,......,n$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)| = \sum_{j=1}^{n} \log\frac{R}{|a_j|} .$$

Lemma 2 is the famous Jensen's Theorem (see page 208 of [1]).

Lemma 3: Let f (z) be analytic for $|z| \le R,\ f(0) \ne 0$ and $|f(z)| \le M$ for $|z| \le R$. Then the number of zeros of f(z) in $|z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{M}{|f(0)|}$.

Lemma 3 is a simple deduction from Lemma 2.

PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda \\
&\quad + ... + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} - (k-1)a_n z^n + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} ...... + (a_{\lambda+1} - \rho a_\lambda)z^{\lambda+1} \\
&\quad + (\rho - 1)a_\lambda z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + ... + (a_1 - a_0)z + a_0
\end{aligned}$$

For $|z| > 1$ so that $\frac{1}{|z|^j} < 1, \forall j = 1,2,......,n$, we have, by using the hypothesis and the Lemma

$$\begin{aligned}
|F(z)| &\ge |a_n z + (k-1)a_n||z|^n - [|ka_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} ..... + |a_{\lambda+1} - \rho a_\lambda||z|^{\lambda+1} \\
&\quad + |\rho - 1||a_\lambda||z|^{\lambda+1} + |a_\lambda - a_{\lambda-1}||z|^\lambda + ... + |a_1 - a_0||z| + |a_0|] \\
&= |z|^n [|a_n z + (k-1)a_n| - \{|ka_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + ... \\
&\quad + \frac{|a_{\lambda+1} - \rho a_\lambda|}{|z|^{n-\lambda-1}} + \frac{(\rho - 1)|a_\lambda|}{|z|^{n-\lambda-1}} + \frac{|a_\lambda - a_{\lambda-1}|}{|z|^{n-\lambda}} + ... + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}] \\
&= |z|^n [|a_n z + (k-1)a_n| - \{|ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + ... \\
&\quad + |a_{\lambda+1} - \rho a_\lambda| + (\rho - 1)|a_\lambda| + |a_\lambda - a_{\lambda-1}| + ... + |a_1 - a_0| + |a_0| \\
&= |z|^n [|a_n z + (k-1)a_n| - \{(|a_{n-1}| - k|a_n|)\cos\alpha + (|a_{n-1}| + k_1|a_n|)\sin\alpha \\
&\quad + (|a_{n-2}| - |a_{n-1}|)\cos\alpha + (|a_{n-2}| + |a_{n-1}|)\sin\alpha + ... \\
&\quad + ... + (\rho|a_\lambda| - |a_{\lambda+1}|)\cos\alpha + (\rho|a_\lambda| + |a_{\lambda+1}|)\sin\alpha + (\rho - 1)|a_\lambda| \\
&\quad + |a_\lambda - a_{\lambda-1}| + ... + |a_1 - a_0| + |a_0|\}] \\
&= |z|^n [|a_n z - (1-k_1)a_n| - \{k|a_n|(\sin\alpha - \cos\alpha) + \rho|\alpha_\lambda|(\cos\alpha + \sin\alpha - 1) \\
&\quad - |\alpha_\lambda| + L + 2\sin\alpha \sum_{j=\lambda+1}^{n-1} |a_j|\}] \\
&> 0
\end{aligned}$$

if

$$\left|a_n z-(1-k)a_n\right| > k\left|a_n\right|(\sin\sim-\cos\sim)+{}^2\left|\sim_{\text{ª}}\right|(\cos\sim+\sin\sim+1))-\left|\sim_{\text{ª}}\right|+L+M\,2\sin\sim\sum_{j=\text{ª}+1}^{n-1}\left|a_j\right|$$

i.e. if

$$\left|z-(1-k)\right| > \frac{1}{\left|a_n\right|}[k\left|a_n\right|(\sin\sim-\cos\sim)+{}^2\left|\sim_{\text{ª}}\right|(\cos\sim+\sin\sim+1))-\left|\sim_{\text{ª}}\right|+L+2\sin\sim\sum_{j=\text{ª}+1}^{n-1}\left|a_j\right|].$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z-(1-k)\right| > \frac{1}{\left|a_n\right|}[k\left|a_n\right|(\sin\sim-\cos\sim)+{}^2\left|\sim_{\text{ª}}\right|(\cos\sim+\sin\sim+1))-\left|\sim_{\text{ª}}\right|+L+2\sin\sim\sum_{j=\text{ª}+1}^{n-1}\left|a_j\right|].$$

Since the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality and since the zeros of P(z) are also the zeros of F(z) , it follows that all the zeros of P(z) lie in

$$\left|z-(1-k)\right| > \frac{1}{\left|a_n\right|}[k\left|a_n\right|(\sin\sim-\cos\sim)+{}^2\left|\sim_{\text{ª}}\right|(\cos\sim+\sin\sim+1))-\left|\sim_{\text{ª}}\right|+L+2\sin\sim\sum_{j=\text{ª}+1}^{n-1}\left|a_j\right|].$$

That proves Theorem 1.

Proof of Theorem 2: Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\ &=-a_n z^{n+1}-(k-1)a_n z^n+(ka_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}......+(a_{\text{ª}+1}-{}^2a_{\text{ª}})z^{\text{ª}+1}\\ &\quad+({}^2-1)a_{\text{ª}}z^{\text{ª}+1}+(a_{\text{ª}}-a_{\text{ª}-1})z^{\text{ª}}+......+(a_1-a_0)z+a_0\ .\\ &=G(z)+a_0\end{aligned}$$

where

$$\begin{aligned}G(z)&=-a_n z^{n+1}-(k-1)a_n z^n+(ka_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}......+(a_{\text{ª}+1}-{}^2a_{\text{ª}})z^{\text{ª}+1}\\ &\quad+({}^2-1)a_{\text{ª}}z^{\text{ª}+1}+(a_{\text{ª}}-a_{\text{ª}-1})z^{\text{ª}}+......+(a_1-a_0)z\ .\end{aligned}$$

For $|z|=R$, we have, by using the hypothesis

$$\begin{aligned}\left|G(z)\right|&\le\left|a_n\right|\left|z\right|^{n+1}+(1-k)\left|a_n\right|\left|z\right|^n+\left|ka_n-a_{n-1}\right|\left|z\right|^n+\left|a_{n-1}-a_{n-2}\right|\left|z\right|^{n-1}......+\left|a_{\text{ª}+1}-{}^2a_{\text{ª}}\right|\left|z\right|^{\text{ª}+1}\\ &\quad+({}^2-1)\left|a_{\text{ª}}\right|\left|z\right|^{\text{ª}+1}+\left|a_{\text{ª}}-a_{\text{ª}-1}\right|\left|z\right|^{\text{ª}}+......+\left|a_1-a_0\right|\left|z\right|\\ &\le\left|a_n\right|R^{n+1}+(1-k)\left|a_n\right|R^n+\left|ka_n-a_{n-1}\right|R^n+\left|a_{n-1}-a_{n-2}\right|R^{n-1}......+\left|a_{\text{ª}+1}-{}^2a_{\text{ª}}\right|R^{\text{ª}+1}\\ &\quad+({}^2-1)\left|a_{\text{ª}}\right|R^{\text{ª}+1}+\left|a_{\text{ª}}-a_{\text{ª}-1}\right|R^{\text{ª}}+......+\left|a_1-a_0\right|R.\end{aligned}$$

Thus, for $R\ge1$, we have, by using the Lemma

$$\begin{aligned}\left|G(z)\right|&\le\left|a_n\right|R^{n+1}+R^n[(1-k)\left|a_n\right|+\left|ka_n-a_{n-1}\right|+\left|a_{n-1}-a_{n-2}\right|+.....+\left|a_{\text{ª}+1}-{}^2a_{\text{ª}}\right|\\ &\quad+({}^2-1)\left|a_{\text{ª}}\right|+\left|a_{\text{ª}}-a_{\text{ª}-1}\right|+......+\left|a_1-a_0\right|]\\ &\le\left|a_n\right|R^{n+1}+R^n[(1-k)\left|a_n\right|+(\left|a_{n-1}\right|-k\left|a_n\right|)\cos\sim+(\left|a_{n-1}\right|+k\left|a_n\right|)\sin\sim\\ &\quad+(\left|a_{n-2}\right|-\left|a_{n-1}\right|)\cos\sim+(\left|a_{n-2}\right|+\left|a_{n-1}\right|)\sin\sim+......+(\left|a_{\text{ª}+1}\right|-{}^2\left|a_{\text{ª}}\right|)\cos\sim\\ &\quad+(\left|a_{\text{ª}+1}\right|+{}^2\left|a_{\text{ª}}\right|)\sin\sim+({}^2-1)\left|a_{\text{ª}}\right|+L-\left|a_0\right|]\\ &\le\left|a_n\right|R^{n+1}+R^n[\left|a_n\right|-k\left|a_n\right|(\cos\sim-\sin\sim+1)-{}^2\left|\sim_{\text{ª}}\right|(\cos\sim-\sin\sim-1)-\left|\sim_{\text{ª}}\right|+L\\ &\qquad-\left|a_0\right|+2\sin\sim\sum_{j=\text{ª}+1}^{n-1}\left|a_j\right|]\\ &=X\end{aligned}$$

and for $R\le1$, we have, by using the Lemma

$$\begin{aligned}\left|G(z)\right|&\le\left|a_n\right|R^{n+1}+R[\left|a_n\right|-k\left|a_n\right|(\cos\sim-\sin\sim+1)-{}^2\left|\sim_{\text{ª}}\right|(\cos\sim-\sin\sim-1)-\left|\sim_{\text{ª}}\right|+L\\ &\qquad+2\sin\sim\sum_{j=\text{ª}+1}^{n-1}\left|a_j\right|]-(1-R)\left|a_0\right|\end{aligned}$$

=Y.

Since G (z) is analytic for $|z| \le R, G(0) = 0$, it follows by Schwarz Lemma that for $|z| \le R$,

$|G(z)| \le X|z|$ for $R \ge 1$ and $|G(z)| \le Y|z|$ for $R \le 1$.

Hence , for $|z| \le R$, $R \ge 1$

$$
\begin{aligned}
|F(z)| &= |a_0 + G(z)| \\
&\ge |a_0| - |G(z)| \\
&\ge |a_0| - X|z| \\
&. > 0
\end{aligned}
$$

if $|z| < \frac{|a_0|}{X}$

and for $R \le 1$

$$|F(z)| > 0$$

if $|z| < \frac{|a_0|}{Y}$.

This shows that F(z) and hence P(z) does not vanish in $|z| < \frac{|a_0|}{X}$ for $R \ge 1$ and in $|z| < \frac{|a_0|}{Y}$ for $R \le 1$. In other words all the zeros of P(z) lie in $|z| \ge \frac{|a_0|}{X}$ for $R \ge 1$ and in $|z| \ge \frac{|a_0|}{Y}$ for $R \le 1$.

Again , for $|z| \le R$, it is easy to see as above that

$$
\begin{aligned}
|F(z)| \le\; & |a_n|R^{n+1} + R^n[|a_n| - k|a_n|(\cos\sim - \sin\sim + 1) - ²|\sim_ª|(\cos\sim - \sin\sim - 1) - |\sim_ª| + L \\
& + 2\sin\sim \sum_{j=ª+1}^{n-1} |a_j|]
\end{aligned}
$$

=A

for $R \ge 1$

and

$$
\begin{aligned}
|F(z)| \le\; & |a_n|R^{n+1} + R[|a_n| - k|a_n|(\cos\sim - \sin\sim + 1) - ²|\sim_ª|(\cos\sim - \sin\sim - 1) - |\sim_ª| + L \\
& + 2\sin\sim \sum_{j=ª+1}^{n-1} |a_j|] - (1-R)|a_0|
\end{aligned}
$$

=B

for $R \le 1$.

Hence , by using Lemma 3, it follows that the number of zeros of F(z) and therefore P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{A}{|a_0|}$ for $R \ge 1$ and the number of zeros of F(z) and therefore P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{B}{|a_0|}$ for $R \le 1$.

That completes the proof of Theorem 2.

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