Available Online at http://www.recentscientific.com

ISSN: 0976-3031

International Journal of Recent Scientific Research
Vol. 7, Issue, 12, pp. 14837-14848, December, 2016

International Journal of Recent Scientific Research

Research Article

PROOF OF UNIQUENESS AND DIFFERENTIABILITY OF THE SOLUTIONS OF COMPRESSIBLE NON STATIONARY DYNAMIC SYSTEMS SOLUTIONS

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ARTICLE INFO

Article History:

Received 05th September, 2016
Received in revised form 21st October, 2016
Accepted 06th November, 2016
Published online 28th December, 2016

Key Words:

Uniqueness and differentiability, compressible dynamic system, inverse functions.

ABSTRACT

In this article, we propose to establish the proof of the differentiability of the solution of a system of nonlinear equations disrupted following a certain number of parameters. Two main results have been derived. In the first theorem, differentiability is defined according to FFrechet.The proof is given using the theorem of reciprocal functions in Banach spaces with prior evidence of Frechet's strict differentiability of adirect application. In the second theorem, differentiability is in a weaker form as to Frechet. Its proof requires the use of the Hadamard theorem of small disturbances of isomorphism in Banach spaces and the theorem of strict differentiability of inverse functions established in[10] (with a possible lack of differentiability in the sense of Fréchet).

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INTRODUCTION

Cancer [2]-[7] is a serious genetic disorder in the number of dead cells as to cells division, leading to cells disequilibrium. The balance between these two processes regulates the number of cells in the tissues, and the breakdown of this equilibrium leads to the development of clusters of cancer cells (called tumors [6]) irrespective of the normal functioning of the body. The cancer cell is a want on cell that multiplies itself in an uncontrolled and excessive manner within a normal tissue of the body. This anarchic proliferation gives rise to increasingly large tumors that grow up and then destroy the surrounding organs.

The cancer cells can also swarm away from a body to form a new tumor, or circulate in a free form. By destroying its environment, the cancer can become a real danger to the survival of an individual.

The fight against this disease is an important field of medical research. The need to adapt various types and forms of cancers as well as the understanding of complex phenomena involved in its growth has led to the development of many mathematical models [3] in recent decades. Mathematical modeling of cancer evolution is a rapidly developing field [13]. Their interest lies in their ability to gather large quantity of information accumulated by biologists. Indeed, it is important to understand that the mathematical complexity of a model is not a sufficient criterion to judge its relevance. Thus, the nature of this phenomenon (*the cancer cells have a fluid-like movement*) motivated us to use the non-stationary compressible Navier-Stokes model, though relatively simple which however describes the disease. These equations do not address the tumor environment and its interactions directly, but present measurable magnitudes such as the volume density denoted by $\circ_v = \circ_v(x,t)$ the density of the outer forces denoted by $\P_e = \P_e(x,t)$, which models environmental factors. Furthermore, it is considered that the cells are transported by a velocity field, say $v = v(x,t)$, withthe related pressure $\circledR = \circledR(x,t)$.

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The choice of the Navier Stokes system as working equations permits to tackle problems like unknown coupling, nonlinearity, and time dependence. The non-linear nature of the convection term $v\nabla v$ that appears in these equations is the source of difficulties in solving this problem. To overcome these challenges, some authors use the method consisting in estimates and weak convergences in regular spaces like $L^2\left(\left[t_i, t_f\right]; L^1(I)\right)$ However, let notice that in this article the goal is to get the differentiability of the compressible system solution $(v, °)$ Navier– Stokes with $v \equiv \mathsf{R}_{£}(V)$, and $V = \left(v_0, °_0, ¶_e\right)$, wherein v_0, $°_0$, and $¶_e$ are respectively the initial velocity, initial density and density function that models the membrane surrounding the tumor and $\Re_{£}$ the operator satisfying $\left\|\Re_{£}(V) - \Re_{£}(\bar{V})\right\| \leq \frac{ª}{\left\|\Psi^{-1}\right\|}\left\|V - \bar{V}\right\|$ where Ψ is a continuous invertible operator.

So our approach is therefore to disrupt our system involving measurable functions and operators and twice continuously differentiable in Banach spaces in order to obtain proof of the differentiability of the solution $(v, °)$.

Problem Formulation

In all fields of biology, the use of mathematics as presentation and forecasting tool has become more and more important nowadays. In this article, we present the tumor problem through a given area. Consider a non-homogeneous region (variable density) depending on the time $I_t = I \times \left(t_0, t_f\right)$ occupied by the tumor, where I is a bounded Lipchitz of either R^3 and either ∂I regular boundary I. We notice x the position occupied by the tumor in $I \subset \mathrm{R}^3$. At the initial instant $t_i = t_0$ the tumor occupies the position x_0 in the space I. The compressible non stationary model is then described by the following (NS_c) equations:

$$\partial_t °v + div(°v \otimes v) - \Lambda v + \nabla ® = °¶_e \qquad (2.1)$$

$$\partial_t ° + div(°v) = 0 \qquad (2.2)$$

With $(x,t) \in I \times \left(t_0, t_f\right)$. Here $¶_e$ denotes the density of the external forces and the operator Λ is defined as $\Lambda v = «\, div(\nabla v) - \left(ª + \frac{«}{3}\right) div(\nabla . v)$ where $ª > 0$ and $« > 0$ respectively represent the bulk viscosity and the dynamic coefficients supposed to be constant. In this system, the pressure is given by the state law $® = k°^{C_a}$, $k \geq 1$ and C_a adiabatic constant as $C_a \geq d/2$. In the following, we set $C_a = 2$.

The system is completed by initial conditions on the volume density and field velocity:

$$°\big|_{t_0=0} = °_0(x)\ v\big|_{t_0=0} = v_0(x) \text{ and } °v\big|_{t_0=0} = q_0(x) \qquad (2.3)$$

It is assumed that on the boundary ∂I the speed satisfies:

$$v\big|_{\partial I} = v(x,t) = 0\ \forall (x,t) \in I \times \left(t_0, t_f\right) \qquad (2.4)$$

It is worth mentioning that $°v \otimes v \in \mathrm{R}^3$ in (NS_c) is a tensor product of $°v$ and v. Then,

$$\nabla.(°v \otimes v) = \nabla.(°v)v + °(v.\nabla)v \qquad (2.5)$$

Notations and approximation of the solution

Notations

Before enouncing the results it is necessary to define the areas in which we work. Let $I \subset R^d$ be a regular bounded Lipchitz border ∂I and let $t \in \left[t_0, t_f\right]$ a sufficiently wide interval. For all $1 \leq p < +\infty$, let $W^{m,p}(I)^d$ represents the usual Sobolev space defined on I and with the norm $\| \, . \, \|_{m,p}$ ($m \geq -1$ denotes an integer).In addition, $L^p(I)^d$ is the Lebesgue space on I with the norm $\| \ \|_p$ while $\| \ \|_{K_2}$ is the norm associated with a given space K_2. If K_1 is a Banach space, we note by $L^p(t_0, t_f; K_1)$ the Banach space consisting of measurable functions on $[t_0, t_f]$ values in K_1.Let $X = L^2\left((t_0, t_f); \left[H_0^1(I)\right]^3\right)$, $Y = L^2\left((t_0, t_f); L^2(I)^3\right)$, $Z = L^2\left((t_0, t_f); L^2(I)\right)$, and $W = L^2\left((t_0, t_f); \left[H^1(I)\right]\right)$.

Approximation of the solution

We will later provide an estimate of the solution v of the (NS_c) problem without the disturbance operator.
Multiplying equation (2.1) by v and integrating over the volume I, we obtain the following variation formula:

$$\forall x \in I \ , \ \forall t \in [t_0, t_f]$$

$$\int_I \text{°} \left(\frac{\partial v}{\partial t}\right) v dx + \int_I \text{°} \left(v \nabla v\right) v dx - \int_I \left(\text{«} \Delta v\right) v dx - \int_I \left(\text{ª} + \frac{\text{«}}{3}\right) \nabla div(v) v dx + \int_I \nabla \text{®} v dx = \int_I \text{°} \text{¶}_e dx$$

By applying the differentiation theorem, the first member of the left gives the following estimate:

$$\int_I \text{°} \left(\frac{\partial v}{\partial t}\right) v dx = \frac{1}{2}\frac{d}{dt}\int_I \text{°} \ \|v\|^2_{H_0^1(I)} dx \ \forall t \in [t_0, t_f] \tag{3.1}$$

The Navier-Stokes equations in slow regime report that the integral over the volume I of the term $(v\nabla v)$ is null due to the assumption of low speed.

$$\int_I \text{°} \left(v \nabla v\right) v dx = 0, \ \forall t \in [t_0, t_f] \tag{3.2}$$

In order to solve the (NS_c) problem, several estimates are required.

Estimate of $\int_I \left(\text{«} \Delta v\right) v dx$.

$$\int_I \left(\text{«} \Delta v\right) v dx = \text{«} \int_{\partial I} \text{¥}_0 v \left(\nabla v . \vec{n}\right) ds - \text{«} \int_I tr\left(\nabla v . \nabla^t v\right) dx$$

(where ¥_0 is a unique continuous linear application definedfrom $W_2^1(I)$ to $L^2(I)$ such that $\text{¥}_0 v = 0$, $\vec{n}$ is the normal to the edge of I,denoted by ∂I and ds its elementary surface increment).It therefore follows that:

$$\int_I \left(\text{«} \Delta v\right) v dx = -\text{«} \sum_{i,j}^{3} \int_I \frac{\partial v_i \partial v_j}{\partial x_i \partial x_j} dx \leq \text{«} \int_I \left\| \frac{Dv}{Dt} \right\|^2 dx \tag{3.3}$$

Estimate of $\int_I \left(\text{ª} + \frac{\text{«}}{3}\right) \nabla div(v) v dx$:

$$\int_I \left(\text{ª} + \frac{\text{«}}{3}\right) \nabla div(v) v dx = \sim \left(\int_I \nabla \left(v div(v) dx - \int_I \Delta v^2 dx \right) \right) \leq \sim \int_I \left\| \frac{Dv}{Dt} \right\|^2 dx \tag{3.4}$$

Where $\sim = \left(\text{ª} + \frac{\text{«}}{3}\right)$.

estimate of $\int_I \nabla \text{®} v dx$

$\int_I \nabla ® v dx = \int_I \nabla k ° ^2 v dx$ after integrating by parts we have:

$$\int_I \nabla k ° ^2 v dx = \int_{\partial I} k ° ¥_0 v.\vec{n} dx - \int_I \frac{\partial}{\partial t}\left(\frac{k ° ^2}{2}\right) dx = -\frac{d}{dt}\int_I k \frac{° ^2}{2} dx \quad (3.5)$$

Finally the force provided by the membrane is: $\forall t \in \left[t_0, t_f\right]$

$$\int_I ° ¶_e v dx \le \| ° v \|_{L^4(I)} \| ¶_e \|_{\left(L^2(I)\right)^3} \quad (3.6)$$

Gathering these different estimates, the expression (2.1) becomes:

$$\frac{1}{2}\frac{d}{dt}\int_I ° \|v\|_{H_0^1(I)^3} dx - \frac{d}{dt}\int_I k \frac{° ^2}{2} dx - (« + \sim)\int_I \left\| \frac{Dv}{Dt} \right\|^2 dx \le \| ° v \|_{L^4(I)} \| ¶_e \|_{\left(L^2(I)\right)^3}$$

$$\frac{1}{2}\frac{d}{dt}\int_I (° \|v\|_{H_0^1(I)^3} - k ° ^2) dx - \left(\frac{4}{3} « + ª\right)\int_I \left\| \frac{Dv}{Dt} \right\|^2 dx \le \| ° v \|_{L^4(I)} \| ¶_e \|_{\left(L^2(I)\right)^3}$$

$$\frac{1}{2}\frac{d}{dt}\int_I (° \|v\|_{H_0^1(I)^3} - k ° ^2 - \left(\frac{8}{3} « + ª\right)\frac{d}{dt} v^2) dx \le \| ° v \|_{L^4(I)^3} \| ¶_e \|_{\left(L^2(I)\right)^3} \quad (3.7)$$

Thus, the solution $(v, °)$ of the problem satisfies the inequality (3.7).

It is of the greatest interest to an estimate of the solution $(v, °)$ under the assumption of low speeds. Hence the following theorem:

Theorem 3.1 *(estimated solution with low speed hypothesis).*

Let $v_0 \in H_0^1(I)^3$, $°_0 \in L^2(I)$, $q_0 \in L^4(I)^3$ and $¶_e \in L^2\left(\left(t_0, t_f\right); L^2(I)^3\right)$. We suppose there exists a constant > 0 such that $\forall (x,t) \in I \times \left(t_0, t_f\right)$, $\left| ° \right|^{-1} =$ and $< °_0$. Then there exists a solution $(v, °)$ of the system (NS_c) satisfying the initial conditions (2.3) and the following inequality:

$$\|v\|_X \le \left[\left(\|q_0\|^2_{L^4(I)^3} + \|¶_e\|^2_Y\right) e^{¢ t}\right]^{1/2} \quad (3.8)$$

Proof

Let ª and « denote the viscosity coefficients assumed to be constant and satisfying physical constraints $\sim = \left(ª + \frac{«}{3}\right)$ and $« > 0$.

From [11] we have the following inequality:

$$\frac{1}{2}\frac{d}{dt}\int_I (° \|v\|_{H_0^1(I)^3} - k ° ^2 - \left(\frac{8}{3} « + ª\right)\frac{d}{dt} v^2) dx \le \| ° v \|_{\left(L^4(I)\right)^3} \| ¶_e \|_{\left(L^2(I)\right)^3}$$

$$\frac{1}{2}\frac{d}{dt}\int_I (° \|v\|_{H_0^1(I)^3}) dx \le \| q \|_{\left(L^4(I)\right)^3} \| ¶_e \|_{\left(L^2(I)\right)^3}$$

By applying the Young inequality, the estimation becomes:

$$\frac{1}{2}\frac{d}{dt}\int_I (\circ \, \|v\|_{H_0^1(I)^3})dx \leq \frac{1}{2}\|q\|^2_{(L^4(I))^3} + \frac{1}{2}\|\P_e\|^2_{(L^2(I))^3}$$

Integrating between t_0 and t_f yields:

$$|\circ|^2\|v\|_X^2 \leq \int_{t_0}^{t_f}\left(\|q\|^2_{(L^4(I))^3}\right)dt + \|q_0\|^2_{(L^4(I))^3} + \int_{t_0}^{t_f}\left(\|\P_e\|^2_{(L^2(I))^2}\right)dt$$

$$|\circ|^2\|v\|_X^2 \leq \int_{t_0}^{t_f}\left(\|q\|^2_{(L^4(I))^3}\right)dt + \|q_0\|^2_{(L^4(I))^3} + \|\P_e\|_Y^2$$

Applying the Gronwall Lemma (see $[4]$)for any $t \geq 0$

$$|\circ|^2\|v\|_X^2 \leq \left(\|q_0\|^2_{(L^4(I))^3} + \|\P_e\|_Y^2\right)\int_{t_0}^{t_f} dt$$

$$\|v\|_X^2 \leq |\circ|^{-2}\left(\|q_0\|^2_{(L^4(I))^3} + \|\P_e\|_Y^2\right)e^{¢t}$$

Under the restricted increase imposed on $|\circ^{-1}|$ in the statement of the theorem, we can establish that

$$\|v\|_X \leq \left[\left(\|q_0\|^2_{L^4(I)^3} + \|\P_e\|_Y^2\right)e^{¢t}\right]^{1/2} \square$$

Linearization system

The characteristics are defined as above, with the same initial conditions and a domain I which is still bounded. We are still interested in studying the system under the assumption of compressibility of cancer cells.

However, let's look at the character $v\nabla v$ that appears in the (2.1).It is at the origin of difficulties when solving this problem. We will linearize this term by substituting the following disturbance:

$$F(H, ¨) = H(x,t) + ¨(x,t,ñ,X) \qquad (4.1)$$

where H is alinear integrableoperator that will bedefined later in the proposition 4.1 and ¨ a function given by:

$$I \times [t_0, t_f] \times {}^3 \times {}^9 \rightarrow [t_0, t_f] \times {}^9 : (x,t,ñ,X) \mapsto ¨(x,t,ñ,X)$$

Then, for all $(x,t) \in I \times (t_0, t_f)$, equation(2.1)becomes :

$$\partial_t \circ v + div(\circ v)v + F(H, ¨) + \nabla ® = \circ \P_e + \Lambda v \qquad (4.2)$$

This approach has introduced new variables, say ñ , X which are considered as a fieldargument $v(x,t)$ and its divergence (describes as anincrease in the volume) respectively.

Proposition4.1: For our study, let consider the functions $¨(x,t,ñ,X)$ and $U(x,t,ñ,X)$ defined on $I \times [t_0, t_f] \times {}^3 \times {}^9$ and satisfying the following assumptions:

Assumptions (H)

H-1: For all $(ñ, X) \in {}^3 \times {}^9$, there exists , $' > 0$ such that the functions

$(x,t,ñ,X) \mapsto ¨(x,t,ñ,X)$ and $(x,t) \mapsto U(x,t,ñ,X)$ are measurable functions and satisfy the following conditions:

$$\left| \ddot{} (x,t,ñ,X) \right| \leq \left(ñ^2 + ¡^2 \right) e^{¢t} \quad (4.3)$$

$$\left| U(x,t,ñ,X) \right| \leq {}' \left(ñ^2 + ¡^2 \right) e^{¢t} \quad (4.4)$$

H-2:For almost all $(x,t) \in I \times [t_0, t_f]$, there exists $\mu, \bar{\mu} > 0$ such that the functions $(x,t,ñ,X) \mapsto \ddot{} (x,t,ñ,X)$ and $(x,t) \mapsto U(x,t,ñ,X)$ are twice continuous and differentiable on ${}^3 \times {}^9$ in addition:

$$\left| \Delta_{ñ} \ddot{} \right| + \left| \Delta_X \ddot{} \right| \leq 4\mu e^{¢t} \text{ and } \left| \Delta_{ñ} U \right| + \left| \Delta_{¡} U \right| \leq 4\bar{\mu} e^{¢t} \quad (4.5)$$

H-3:let $H \equiv P u$ be a continuous linear integral operator, for which any function u correspond to H such that:

$$H_p u(.,t) := \int_I \int_{t_0}^{t_f} P(x,t,y) u(y,t) \, dt \, dy$$ is defined by:

$$H : L^2(I) \times [t_0, t_f] \mapsto L^2(I) \times [t_0, t_f] \quad (4.6)$$

H-4:let $A_£$ and $T_£'$ be two non-linear differentiable operators in $L^2\left((t_0, t_f) \times W_2^1(I)\right)$

We have the following formulas:

$$d\left[A_1'(v) g, h\right]_{h=g} = \sum_{i=1}^{3} \partial_X^2 \ddot{} \frac{\partial^2 g^2}{\partial x \partial t} + \partial_{ñ}^2 \ddot{} g^2 \text{ and} \quad (4.7)$$

$$d\left[T_£'(v) g, h\right]_{h=g} = \sum_{i=1}^{3} \partial_X^2 U \frac{\partial^2 g^2}{\partial x \partial t} + \partial_{ñ}^2 U g^2 \quad (4.8)$$

Study of strict ε-differentiability

In this section, let I_p be the disruption of domain I and define a displacement field of Ω defined from ${}^3 \rightarrow {}^3$

$$I_p = \{(x,t) \in I_t, v \in \Omega, Id + ² \Omega\}$$

Definition5.1

Let E_1 and E_2 , two normed spaces I an open set in E_1. Let $M_£$ all compact systems E_1.

If $J'(v+g)h - J'(v)h = J''(v)h^2 + J\left(\|h^2\|\right)$ or $J''(v)h^2 \in L(E_1, L(E_1, E_2))$ with $J''(v)h^2$ is a bilinear operator. The function $J: I \rightarrow E_2$ is called strictly ε-differentiable on I if the condition $(D_£)$ is satisfied

$$(D_£): \left\{ \begin{pmatrix} \forall ¦ > 0, \forall h \in M_£, \forall v \in I, \exists ª > 0 \\ \|v - v_f\| < ª, |d| < ª, v + th^2 \in I \end{pmatrix} \Rightarrow \left\| J\left(h^2\right) \right\| \leq ¦ \, |d| \right\}$$

Proposition 5.2. Let I_p a disturbed areaof I defined as follows:

$$I_p = \{(x,t) \in I_t, v \in \Omega, v + ² \Omega\}$$

The operator $\ddot{}$ is a ε-continuousand ε- differentiable on X .

Proof

Suppose that $A_£$ is Frechet-differentiable and v it a first variation, that is $\lim_{² \rightarrow 0} \frac{A_£(v + ² g) - A_£(v)}{²} = ¢A_£(v, g)$

It is therefore clear that for all $g \in X$, the quantity $A_{£}\left(v,^{2} g\right)$ is defined for ^{2} small enough. After we suppose that $¢ A_{£}\left(v,g\right) = A_{£}^{'}\left(v\right) g$

Let show that $A_{£}$ is twice- differentiable function according to Gateau X.

Assume that $A_{£}$ is Fréchet differentiable We have for all $\left|^{2}\right|$ small enoughand for all $g \in X$

$$A_{£}\left(v+g\right) - A_{£}\left(v\right) = d A_{£}\left(v+g\right) + O\left(g\right)$$

For $^{2} \in \left]-1,1\right[\quad ,^{2} \neq 0$, $A_{£}\left(v+^{2} g\right) - A_{£}\left(v\right) = ¢ A_{£}\left(v+^{2} g\right) + O\left(^{2} g\right)$

We have $A_{£}^{'}\left(v+g\right)h - A_{£}^{'}\left(v\right)h - d\left[A_{£}^{'}\left(v\right)g,h\right]_{h=g} = A_{£}^{'}\left(v+g\right)h - A_{£}^{'}\left(v\right)h - A_{£}^{''}\left(v\right)h^{2}$

Taking the $L^{2}-\text{norm}$ in X, we have:

$$\left\| \frac{A_{£}^{'}\left(v+^{2} g\right)h - A_{£}^{'}\left(v\right)h}{2} - A_{£}^{''}\left(v\right)h^{2} \right\|_{X}^{2} = \left\| \frac{A_{£}^{'}\left(v+^{2} g\right)h}{2} - A_{£}^{''}\left(v\right)h^{2} - \frac{A_{£}^{'}\left(v\right)h}{2} \right\|_{X}^{2}$$

$$= \left\| \frac{A_{£}^{'}\left(x,t,v+^{2} g,\nabla v+^{2}\nabla g\right)h}{2} - \partial_{ñ}^{2\cdot\cdot} h^{2} - \sum_{i}^{3} \partial_{X}^{2\cdot\cdot} \frac{\partial^{2} h^{2}}{\partial x \partial t} - \frac{A_{£}^{'}\left(x,t,v,\nabla v\right)h}{2} \right\|_{X}^{2}$$

$$= \left\| \frac{A_{£}^{'}\left(x,t,v+^{2} g,\nabla v+^{2} g\right)h}{2} - \frac{A_{£}^{'}\left(x,t,v,\nabla v+^{2}\nabla g\right)h}{2} - \partial_{ñ}^{2\cdot\cdot} h^{2} + \right.$$

$$\left. \frac{A_{£}^{'}\left(x,t,v,\nabla v+^{2}\nabla g\right)h}{2} - \frac{A_{£}^{'}\left(x,t,v,\nabla v\right)h}{2} - \sum_{i}^{3} \partial_{X}^{2\cdot\cdot} \frac{\partial^{2} h^{2}}{\partial x \partial t} \right\|_{X}^{2}$$

$$\leq \int_{I}\int_{t_0}^{t_f} \left(\left| \frac{A_{£}^{'}\left(x,t,v+^{2} g,\nabla v+^{2}\nabla g\right)h}{2} - \frac{A_{£}^{'}\left(x,t,v,\nabla v+^{2}\nabla g\right)h}{2} - \partial_{ñ}^{2\cdot\cdot} h^{2} \right|^{2} \right) dtdx +$$

$$\int_{I}\int_{t_0}^{t_f} \left(\left| \frac{A_{£}^{'}\left(x,t,v,\nabla v+^{2}\nabla g\right)h}{2} - \frac{A_{£}^{'}\left(x,t,v,\nabla v\right)h}{2} - \sum_{i}^{3} \partial_{X}^{2\cdot\cdot} \frac{\partial^{2} h^{2}}{\partial x \partial t} \right|^{2} \right) dtdx$$

Using Lagrange's formula for some $^{-} \in \left[0;1\right]$

$$\leq \int_{I}\int_{t_0}^{t_f} \left(\left| \int_{0}^{1} A_{£}^{''}\left(x,t,v+^{-2} g,\nabla v+^{2}\nabla g\right)h^{2} - \partial_{ñ}^{2\cdot\cdot} h^{2} \right|^{2} d^{-} \right) dtdx +$$

$$\int_{I}\int_{t_0}^{t_f} \left(\left| \int_{0}^{1} A_{£}^{''}\left(x,t,v,\nabla v+^{-2}\nabla g\right)h^{2} - \sum_{i}^{3} \partial_{X}^{2\cdot\cdot} \frac{\partial^{2} h^{2}}{\partial x \partial t} \right|^{2} d^{-} \right) dtdx$$

$$\leq \int_{I}\int_{t_0}^{t_f} \left(\left| \int_{0}^{1} \left(\left(\partial_{ñ}^{2\cdot\cdot} \left(x,t,v+^{-2} g,\nabla v+^{2}\nabla g\right)h^{2} - \partial_{ñ}^{2\cdot\cdot} h^{2} \right) \times \sum_{i} \partial_{X}^{2\cdot\cdot} \frac{\partial^{2} h^{2}}{\partial x \partial t} \right|^{2} d^{-} \right) \right) dtdx +$$

$$\int_I\int_{t_0}^{t_f}\left(\left|\int_0^1\sum_i\left(\partial_X^2\,(x,t,\nabla v+\varepsilon\nabla g)h^2-\partial_X^2\,h^2\right)\times\sum_i\partial_X^2\,\frac{\partial^2h^2}{\partial x\partial t}\right|^2 d\theta\right)dtdx$$

$$\leq\int_0^1\left(\int_I\int_{t_0}^{t_f}\left(\left|\partial_\rho^2\,(x,t,v+\theta\varepsilon g,\nabla v+\varepsilon\nabla g)h^2-\partial_X^2\,h^2\right|^2\times\left|\sum_i^3\partial_X^2\,\frac{\partial^2h^2}{\partial x\partial t}\right|^2 dtdx\right)d\theta+$$

$$\int_0^1\left(\int_I\int_{t_0}^{t_f}\left(\sum_i^3\left|\partial_\rho^2\,(x,t,v,\nabla v+\varepsilon\nabla g)h^2-\partial_X^2\,h^2\right|^2\times\left|\sum_i^3\partial_X^2\,\frac{\partial^2h^2}{\partial x\partial t}\right|^2 dtdx\right)d\theta$$

From Newton-Leibnizformula, Cauchy inequality and using (4.4),we obtain

$$\lim_{\varepsilon\to0}\left\langle d\left[A_1'(v)h,g\right]_{h=g},h\right\rangle=\quad o\left(\|g\|^2\right)$$

On the other hand, let $m\in[0;f]$ suppose that there exists a sequence v_m of X such that for all integer m

We have: $\begin{cases}v_m\to v_f & \text{in } X\\ \nabla v_m\to\nabla v_f & \text{in } Y\end{cases}$

Then there exists $h\in X$ such that $d\left[A_1'(.)h,g\right]_{h=g}\notin L^2$ spaceso that $\left\|d\left[A_1'(v_m)h,g\right]_{h=g}-d\left[A_1'(v_f)h,g\right]_{h=g}\right\|_X^2\neq0$ for $m\to0$ thus there exists $\sim\ \geq1$ such that $\left\|d\left[A_1'(v_m)h,g\right]_{h=g}-d\left[A_1'(v_f)h,g\right]_{h=g}\right\|_X^2\geq\frac{\sim}{2}$

Indeed $\left\|d\left[A_1'(v_m)h,g\right]_{h=g}-d\left[A_1'(v_f)h,g\right]_{h=g}\right\|_X^2=$

$$\left\|\partial_X^2\,(x,t,v_m,\nabla v)h^2+\sum_{i=1}^3\partial_\rho^2\,(x,t,v_m,\nabla v_m)\frac{\partial^2h^2}{\partial x\partial t}-\partial_X^2\,(x,t,v_f,\nabla v_f)h^2+\sum_{i=1}^3\partial_\rho^2\,(x,t,v_f,\nabla v_f)\frac{\partial^2h^2}{\partial x\partial t}\right\|_X^2$$

$$\leq\int_{t_0}^{t_f}\int_I\left|\partial_X^2\,(x,t,v_m,\nabla v_m)-\partial_X^2\,(x,t,v_f,\nabla v_f)h^2\right|^2 dtdx+$$

$$\int_{t_0}^{t_f}\int_I\left|\sum_i\partial_\rho^2\,(x,t,v_m,\nabla v_m)-\partial_\rho^2\,(x,t,v_f,\nabla v_f)\frac{\partial^2h^2}{\partial x\partial t}\right|^2 dtdx$$

$$\leq\int_{t_0}^{t_f}\int_I\sim\left|\partial_\rho^2\,(x,t,v_m,\nabla v_m)-\partial_\rho^2\,(x,t,v_f,\nabla v_f)\right|^2\left|h^2\right|dtdx+$$

$$\int_{t_0}^{t_f}\int_I\sim\left|\sum_i\partial_\rho^2\,(x,t,v_m,\nabla v_m)-\partial_\rho^2\,(x,t,v_f,\nabla v_f)\right|^2\left|\frac{\partial^2h^2}{\partial x\partial t}\right|^2 dxdt$$

$$\leq16\sim\mu^2e^{\phi t}\int_{t_0}^{t_f}\left|h^2\right|dt+24\sim\mu^2e^{\phi t}\int_{t_0}^{t_f}\left|\frac{\partial^2h^2}{\partial x\partial t}\right|dt$$

$$\leq40\sim\mu^2e^{\phi t}\|h\|_X^2$$

According to the H-2 hypothesis for all $m \in [1\ ;f], v_m \to v_f$ et $\nabla v_m \to \nabla v_f$ pp. in Y

$$\sim \left|\partial_X^2 \cdot\cdot (x,t,v_m,\nabla v_m) - \partial_X^2 \cdot\cdot (x,t,v_f,\nabla v_f)\right|^2 |h|^2 \to 0 \text{ in the same way}$$

$$\sim \left|\sum_i \partial_{\tilde{n}}^{2\cdot\cdot} (x,t,v_m,\nabla v_m) - \partial_{\tilde{n}}^{2\cdot\cdot} (x,t,v_f,\nabla v_f)\right|^2 \left|\frac{\partial^2 h^2}{\partial x \partial t}\right|^2 \to 0$$

Using double integration, iwe obtain:

$$\left\| d\left[A'_{£}(v_m)h,g\right]_{g=h} - d\left[A'_{£}(v_f)h,g\right]_{g=h} \right\|_X^2 \to 0 \text{ for } m \to f$$ Which contradicts our hypothesis.

However, it was therefore $d\left[A'_{£}(.)h,g\right]_{g=h}$ belongs to the space $(X;Y)$

We can therefore conclude that the second variation of the operator $A_{£}$ equals $d\left[A'_{£}(.)h,g\right]_{g=h}$ $\forall v,h \in X$ and for a given speed $v(x,t)$, $d\left[A'_{£}(.)h,g\right]_{g=h}$ in general it will be a linear operator space $E_{ep}(X;Y)$.However, according to the above we can say that $A'_{£}$ is ε-continuousand ε-differentiable on X .

Proposition 5.3 Let I tobe a bounded open set in ^{3}.

Let $\ell(x,t) \in X$ and $h(x,t) \in X$, $\forall n$ there exists $d_n > 0$ such that for $2^n \in]0;1[$ we have :

$|2^n| < d_n$, and h_n is a small enough such that $\|h_n\|_X \leq 1$, then

$$\left|\left\langle \frac{1}{2^n} J_{£}\left(\left\|2^n h_n^2\right\|\right), \cdot\ (x,t)\right\rangle\right| \leq |2^n| \text{ for } \|v_n - v_f\| \to 0,$$

(We say That $v_n(x,.) \to v_f(x,t_f)$ almost over I_t)

Proof:

Let $v_f, v_n \in H_0^1(I)^3$ such that for $n \in [1;f]$, $v_n \to v_f$ pp. in $I \times [t_0;t_f]$

Let h_n be small enough as $\|h_n\|_X \leq 1$

Let $J_{£}\left(\left\|h_n^2\right\|\right) = A'_{£}(v+h_n)h_n - A'_{£}(v)h_n - A''_{£}(v_f)h_n^2$ such thatfor $2^n \in]0;1[$,

$$J_{£}\left(\left\|2^n h_n^2\right\|\right) = A'_{£}(v+2^n h_n)h_n - A'_{£}(v)h_n - A''_{£}(v_f)2^n h_n^2$$

We get $\left|\left\langle \frac{1}{2^n} J_{£}\left(\left\|2^n h_n^2\right\|\right), \cdot\ (x,t)\right\rangle\right| = \left|\int_I \int_{t_0}^{t_f} \frac{1}{2^n} J_{£}\left(\left\|2^n h_n^2\right\|\right) \times \cdot\ (x,t)\, dt dx\right| =$

$$=\left|\int_{t_0}^{t_f}\int_Q \frac{1}{2^n}\left[\partial_{\tilde{n}}^{\cdot\cdot}\left(x,t,v_n+2^n h_n,\nabla v_n+2^n\nabla v_n\right)h_n-\partial_{\tilde{n}}^{\cdot\cdot}\left(x,t,v_n,\nabla v_n\right)h_n-\partial_{\tilde{n}}^{2\cdot\cdot}\left(x,t,v_f\right)2^n h_n^2\right]\ell(x,t)\,dtdx\right|$$

So from Lagrange's formula $[11]$ fora some $^{-}\in[0;1]$, the equality becomes:

$$=\left|\int_{t_0}^{t_f}\int_I \frac{1}{2^n}\left[\int_0^1\left(\partial_{\tilde{n}}^{2\cdot\cdot}\left(x,t,v_n+{}^{-}2^n h_n\right)2^n h_n^2-\partial_{\tilde{n}}^{2\cdot\cdot}\left(x,t,v_f\right)2^n h_n^2 d^{-}\right]\times\cdot\ (x,t)\,dtdx\right|$$

$$=\left|\int_{t_0}^{t_f}\int_I \frac{1}{2^n}\left[\int_0^1\partial_{\tilde{n}}^{2\cdot\cdot}\left(x,t,v_n+{}^{-}2^n h_n\right)-\partial_{\tilde{n}}^{2\cdot\cdot}\left(x,t,v_f\right)d^{-}\right]2^n h_n^2\times\cdot\ (x,t)\,dtdx\right|$$

From Cauchy Schwarz inequality, we deduce that

$$\leq\int_{t_0}^{t_f}\int_I\left(\left|\int_0^1\left(\partial_{\tilde{n}}^{2\cdot\cdot}\left(x,t,v_n+{}^{-}2^n h_n\right)-\partial_{\tilde{n}}^{2\cdot\cdot}\left(x,t,v_f\right)\right)d^{-}\right|^2 h_n^2\,dtdx\right)^{1/2}\times\left(\int_{t_0}^{t_f}\int_I \cdot^{\,2}(x,t)\,dtdx\right)^{1/2}$$

$$\leq\int_{t_0}^{t_f}\int_I\left(\left|\int_0^1\left(\partial_{\tilde{n}}^{2\cdot\cdot}\left(x,t,v_n+{}^{-}2^n h_n\right)-\partial_{\tilde{n}}^{2\cdot\cdot}\left(x,t,v_f\right)\right)d^{-}\right|^2 dtdx\right)^{1/2}\times\left(\int_{t_0}^{t_f}\int_I \cdot^{\,2}(x,t)\,dtdx\right)^{1/2}\|h_n\|_X$$

On the other hand, $\|v_n-v_f\|\to 0$ and $2^n h_n\to 0$ pp. in $I\times[t_0;t_f]$ then

$\left(\Delta_{\tilde{n}}^{\cdot\cdot}\left(x,t,v_n+{}^{-}2^n h_n\right)-\Delta_{\tilde{n}}^{\cdot\cdot}\left(x,t,v_f\right)\to 0.\right.$

This ends the proof.□

Proposition 5.3. Let I a bounded Lipchitz open interval in ^{3}.And Let $\hat{v}\in X$ Such that ∇v and $\nabla\hat{v}\in Z$ .Let g be small enough such that $\|g\|_X\leq 1$

Suppose that the operator ∇ at any point of $I\times[t_0;t_f]$ satisfies the following inequality:

$$\|\nabla v(x,t)-\nabla v^*(x,t)\|_Z\leq k\|v(x,t)-v^*(x,t)\|_X \quad (5.3)$$

Then for $k,\mu>0$, the operator $A_£$ satisfies :

$$\|A'_£(v)(x,t)-A'_£(v^*)(x,t)\|_W\leq 4\mu(k+1)e^{¢t}\|v-v^*\|_X \quad (5.4)$$

Proof:

$$\|A'_£(v)-A'_£(\hat{v})\|_W=\left\|\sum_{i=1}^3\partial_X^{\cdot\cdot}(v,\nabla v)\frac{\partial^2 g}{\partial x\partial t}+\partial_{\tilde{n}}^{\cdot\cdot}(v,\nabla v)g-\sum_{i=1}^3\partial_X^{\cdot\cdot}(\hat{v},\nabla\hat{v})\frac{\partial^2 g}{\partial x\partial t}-\partial_{\tilde{n}}^{\cdot\cdot}(\hat{v},\nabla\hat{v})g\right\|_W$$

$$\leq \left\| \partial_{ñ} {}^{\cdot\cdot} (v,\nabla v) g + \sum_{i=1}^{3} \partial_{X} {}^{\cdot\cdot} (v,\nabla v) \frac{\partial^2 g}{\partial x \partial t} - \partial_{ñ} {}^{\cdot\cdot} (\hat{v},\nabla v) g - \sum_{i=1}^{3} \partial_{X} {}^{\cdot\cdot} (\hat{v},\nabla v) \frac{\partial^2 g}{\partial x \partial t} \right\|$$

$$+ \left\| \partial_{ñ} {}^{\cdot\cdot} (\hat{v},\nabla v) g + \sum_{i=1}^{3} \partial_{X} {}^{\cdot\cdot} (\hat{v},\nabla v) \frac{\partial^2 g}{\partial x \partial t} - \sum_{i=1}^{3} \partial_{X} {}^{\cdot\cdot} (\hat{v},\nabla \hat{v}) \frac{\partial^2 g}{\partial x \partial t} - \partial_{ñ} {}^{\cdot\cdot} (\hat{v},\nabla \hat{v}) g \right\|$$

$$\leq \left\| \partial_{ñ} {}^{\cdot\cdot} (v,\nabla v) g - \partial_{ñ} {}^{\cdot\cdot} (\hat{v},\nabla v) g \right\| + \left\| \sum_{i=1}^{3} \partial_{X} {}^{\cdot\cdot} (v,\nabla v) \frac{\partial^2 g}{\partial x \partial t} - \sum_{i=1}^{3} \partial_{X} {}^{\cdot\cdot} (\hat{v},\nabla v) \frac{\partial^2 g}{\partial x \partial t} \right\|$$

$$+ \left\| \partial_{ñ} {}^{\cdot\cdot} (\hat{v},\nabla v) g - \partial_{ñ} {}^{\cdot\cdot} (\hat{v},\nabla \hat{v}) g \right\| + \left\| \sum_{i=1}^{3} \partial_{X} {}^{\cdot\cdot} (\hat{v},\nabla v) \frac{\partial^2 g}{\partial x \partial t} - \sum_{i=1}^{3} \partial_{X} {}^{\cdot\cdot} (\hat{v},\nabla \hat{v}) \frac{\partial^2 g}{\partial x \partial t} \right\|$$

$$\leq \left[\int_{t_0}^{t_f} \int_I \left| \partial_{ñ} {}^{\cdot\cdot} (v,\nabla v) - \partial_{ñ} {}^{\cdot\cdot} (\hat{v},\nabla v) \right|^2 dtdx \right]^{1/2} + \left[\int_{t_0}^{t_f} \int_I \left| \partial_{ñ} {}^{\cdot\cdot} (\hat{v},\nabla v) - \partial_{ñ} {}^{\cdot\cdot} (\hat{v},\nabla \hat{v}) \right|^2 dtdx \right]^{1/2}$$

$$\leq \left[\int_{t_0}^{t_f} \int_I 16 \mu^2 e^{2¢t} \left| (v) - (\hat{v}) \right|^2 dtdx \right]^{1/2} + \left[\int_{t_0}^{t_f} \int_I 16 \mu^2 e^{2¢t} \left| (\nabla v) - (\nabla \hat{v}) \right|^2 dtdx \right]^{1/2}$$

$$\leq 4\mu e^{¢t} \left[\left\| v(x,t) - \hat{v}(x,t) \right\|_X + \left\| \nabla v(x,t) - \nabla \hat{v}(x,t) \right\|_X \right] \left\| A_{£}(v) - A_{£}(\hat{v}) \right\|_W \leq 4\mu e^{¢t} (k+1) \left\| v(x,t) - \hat{v}(x,t) \right\|_X$$

Remark5.3in the same way, we can also show that, the operator $T_{£}$ satisfies the inequality (5.4) .On the other hand, P and $T_{£}$ are two continuous linear applications and from the Proposition 5.3, the operator $P\left[T_{£}(v)\right]$ is also Lipchitz. Indeed if we take $A_{\mathbf{v}} = A_{£}(v) + P\left[T_{£}(v)\right]$, we simply show that

$$\left\| A_{£}(v) + P\left[T_{£}(v)\right] - A_{£}(\hat{v}) - P\left[T(\hat{v})\right] \right\|_W \leq c \max(\mu, ´) \left[4e^{¢t}(k+1)\right] \left\| v - \hat{v} \right\|_X \quad (5.5)$$

Theorem5.4

Assume that the initial terms $(2.3),(2.4)$ and assumptions **H-1, H-2**on $^{\cdot\cdot}$ and U are satisfies.

Suppose that there exists a real $¥ > -1$ such that for all b , with $0 < b < ¥$ or $b = \max_{j=1;2} \left| 4\mu_j e^{¢t} \right|$ then there exists a time

$t_f \in \left[t_0; t_f\right]$ and an unique solution $v = R_{£}(v_0, °_0, ¶_e)$ of the problem (4.1) for all $v_0 \in H_0^1(Q)^3$, $°_0 \in L^4(Q)$, $\cdot_e \in Y$

More: $H_0^1(Q)^3 \times L^4(Q) \times Y \;\; \rightarrow \;\; X$

$(v_0, °_0, ¶_e) \mapsto \Re_{£}(v_0, °_0, ¶_e)$ is ε-continuous and ε-differentiable.

On the other hand the operator $R_{£}$ is strongly differentiable on $H_0^1(Q)^3 \times L^4(Q) \times Y$ as an application on the space $(X;\pm)$ and a σ- weak topology in X .

Proof

Consider Q a subspace of X : $Q := \left\{ v \in X, \; \exists ¶_e \in Y, \exists v_0 \in H_0^1(I)^3 \text{ et } °_0 \in L^4(I)^3 \text{ such that } Lv := (v_0, °_0, ¶_e) \right\}$

Let $Z\,v$ an operator defined from the condition (2.3)

$$Zv \;\;:\;\; Q \rightarrow Y \times H_0^1(Q)^3 \times L^4(Q)$$

$v \mapsto (Lv, v_0, °_0)$

Using the norm on Q , we how that Lv is linear, continuous and has an inverse which is also continuous, more $\|Z\ v\|^{-1} \leq \frac{1}{4\mu exp(t_f)}$

Furthermore, if the inequality (5.4) and (5.5) are satisfied, $Z\ \mathbf{v}$ is continuous and reversible, more, $A_{\mathbf{v}}$ is Lipchitz, then using Hadamard theorem, we can write that for all $v_0 \in Q$. The operator $\Re(v) \equiv \left(Lv + d\left[A'_{£}(v_0)h, g \right]_{g=h} + \int_{t_0}^{t_f}\int_I \mathrm{P}\, d\left[\mathrm{T}'_{£}(v)g, h \right]_{h=g} dtdx\ , v_0, °_0 \right): Q \to Y \times H_0^1(I)^3 \times L^4(I)$ has a continuous inverse function in the following form: $\Re(v)^{-1} \equiv \left(Lv + \left[A'_{£}(v)h, g \right]_{g=h} + \int_{t_0}^{t_f}\int_I \mathrm{P}\left[\mathrm{T}'_{£}(v)g, h \right]_{h=g} dtdx\ , v_0, °_0 \right):$

$Y \times H_0^1(I)^3 \times L^4(I) \to Q$.

$\Re_{£}(v)^{-1}$ has an inverse Lipchitz.function, then there is an unique solution $v \equiv \mathsf{R}_{£}(V)$. However according to the Proposition 5.3, $\Re_{£}(v)^{-1}$ isε-strongly differentiablefunction, then for all $v_0 \in Q$ obtained by the strong theorems of differentiable function that $\Re_{£}$ is s-continuous ands- differentiable function and $(X;\pm)$ is strongly differentiable onspace $Y \times H_0^1(I)^3 \times L^4(I)$. □

CONCLUSION

In this paper, we have presented a mathematical model in the third order time for a given tumor, the model based on Navier–Stokes equations with some initial parameters and conditions.

An estimate is given for the speed v of cancer cells with which it grows and spreads (to determine a tumor's rate of growth and spread).

The addition of linear members to our first system allowed us therefore to find a field in which we could solve this problem. However, the results obtained can be used in the theory of optimal command, to establish the necessary optimality conditions, where differentiable functional solution depending on parameters, as well as applications that are part of the type of constraints: equalities and inequalities, which it value as being determined.

References

1. Baillet. Cancérologie Niveau DCEM3 Université Pierre et Marie Curie2002 –2003.
2. Ginestier C., Korkaya H., Dontu G., Birnbaum D. Wicha M.S., Charafe- Jauffret E.(2007). La Cellule Souche Cancéreuse. Médecine/Sciences, 23, 1133-1139.
3. Gireg Desmeulles. Réification des interactions pour l'expérience in virtuo de systèmes biologiques multi-modèles. https://tel.archives-ouvertes.fr/tel-00142697
4. H. Brezis, *Analyse fonctionnelle*, édition Masson, (1993).
5. J.E.Marsden, A.Mathematical, Introduction to. Fluid Mechanics Springer-Verlag, 3rd.ed.New York, 1992.
6. J.F. HERON .Histoire générale du cancer Faculté de Médecine de Caen – France
7. Johanne Marcotte et Renée Ouimet. Le cancer: les nouvelles connaissances usuelles. Bibliothèque nationale du Québec. ISBN : 2-922908-10-0
8. J.P.Aubin, Initiation à l’Analyse Appliquée, Masson, Paris, 1994.
9. Patrick AMAR. Contributions à l’étude de la dynamique des systèmes biologiques et aux systèmes de calcul en biologie synthétique. Université Paris Sud.
10. Souhinine M. F. Théorème sur les fonctions implicites dans les espaces linéaires topologiques. Résumé de la thèse. Université d’Etat de Moscou, 973. –18 p.
11. V. Trenoguine, *Analyse fonctionnelle*, Éditions Mir-Moscou, (1985).
12. Al-Hajj M, Wicha MS, ito-Hernandez A, *et al.* Prospective identification of tumorigenic breast cancer cells. *ProcNatlAcadSci USA* 2003; 100: 3983–3988.
13. D. Wodarz and N. L. Komarova, Dynamics of Cancer: Mathematical Foundations of Oncology (World Scientific, New Jersey, 2014). [5] S. H. Moolgavkar and A. G. Knudson, J. Natl. Cancer Inst. 66