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Research Article

APPLICATION OF FRACTION CALCULUS AND OTHER POPERTIES OF UNIVALENT FUNCTIONS ASSOCIATED WITH SUBORDINATION

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ABSTRACT

There are many sub classes of univalent functions. The objectives of this paper is to introduce new classes and we have attempted to obtain Application of Fraction Calculus and Other Properties for the classes $\mathcal{H}(A,B,\alpha)$ and $K\mathcal{H}(A,B,\alpha)$

INTRODUCTION

Let Tdenote the class of functions $f(z)$ of the form

$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0$ ………(1)

which are univalent in the unit disc $U = \{z : z \in \mathbb{C} and \mid z \mid < 1\}$

Definition 1.1 :A function $f(z) \in T$ is said to be close to convex of order $\mu\ (0 \leq \mu < 1)$ if $Re\,\{f'(z)\} > \mu$ for all $z \in U$

A function $f(z) \in T$ is said to be in the subclass $H(\mu)$ of starlike function if $Re\left(\frac{zf'(z)}{f(z)}\right) > \mu,\ z \in U\ \ 0 \leq \mu < 1$

Definition 1.2 : A function $f(z) \in T$ is said to be in the subclass $G(\mu)$ of convex function if

$$Re\left(1 + \frac{zf'(z)}{f(z)}\right) > \mu, \qquad z \in U$$

Definition 1.3: Let$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0, b_k \geq 0$ then the convolution is defined as

$$f(z) * g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k \qquad \text{... ... (1.3.1)}$$

Definition 1.4 : If f and g are regular in U , we say that f is subordinate to g , denoted by $f \prec g$ or $f(z) \prec g(z)$, if there exist a Schwarz function w, which is regular in U with $w(0) = 0$ and $|w(z)| < 1$
$z \in U$such that $f(z) = g(w(z))$, $z \in U$. In particular if g is univalent in U , we have the equivalence
$f(z) \prec g(z)$if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$

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Definition 1.5 : We say that a function $f(z) \in T$ is in the class $\mathcal{H}(A,B,\alpha)$ if it satisfy

$$\frac{zf'(z)+\alpha z^2 f''(z)}{\alpha zf'(z)+(1-\alpha)f(z)} \prec \frac{1+Az}{1+Bz} \qquad \ldots\ldots (1.5.1)$$

for $0<\alpha\leq 1$, $-1\leq B<A\leq 1$

Furthermore a function $f(z) \in T$ is said to belong to the class $K\mathcal{H}(A,B,\alpha)$ if and only if $zf'(z) \in \mathcal{H}(A,B,\alpha)$.

THEOREM 1.1:A function $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \geq 0$ is in $\mathcal{H}(A,B,\alpha)$ if and only if

$$\sum_{k=2}^{\infty}\{k+\alpha k^2-2\alpha k-(1-\alpha)-[B(k+\alpha k(k-1))-A(\alpha k+1-\alpha)]\}a_k \leq (A-B)$$

PROOF: Suppose $f(z)$ is in $\mathcal{H}(A,B,\alpha)$
Therefore from(1.5.1) we have

$$p(z)=\frac{zf'(z)+\alpha z^2 f''(z)}{\alpha zf'(z)+(1-\alpha)f(z)} \prec \frac{1+Az}{1+Bz}$$

$$p(z)=\frac{1+Aw(z)}{1+Bw(z)}$$

$|w(z)|<1$

$$\left|\frac{\left[\frac{zf'(z)+\alpha z^2 f''(z)}{\alpha zf'(z)+(1-\alpha)f(z)}\right]-1}{A-B\left\{\frac{zf'(z)+\alpha z^2 f''(z)}{\alpha zf'(z)+(1-\alpha)f(z)}\right\}}\right|<1$$

$$\left|\frac{zf'(z)+\alpha z^2 f''(z)-\alpha zf'(z)-(1-\alpha)f(z)}{A[\alpha zf'(z)+(1-\alpha)f(z)]-B[zf'(z)+\alpha z^2 f''(z)]}\right|<1 \qquad \ldots\ldots\ldots\ldots\ldots\mathbf{(1.1)}$$

$$zf'(z)+\alpha z^2 f''(z)-\alpha zf'(z)-(1-\alpha)f(z)$$

$$= -\sum_{k=2}^{\infty}[k+\alpha k^2-2\alpha k-(1-\alpha)]\,a_k z^k$$

$$A[\alpha zf'(z)+(1-\alpha)f(z)]-B[zf'(z)+\alpha z^2 f''(z)]$$

$$=(A-B)z+\sum_{k=2}^{\infty}[-A(\alpha k+1-\alpha)+B(k+\alpha k(k-1))]\,a_k z^k$$

From (1.1) we have

$$\left|\frac{-\sum_{k=2}^{\infty}[k+\alpha k^2-2\alpha k-(1-\alpha)]\,a_k z^k}{(A-B)z+\sum_{k=2}^{\infty}[-A(\alpha k+1-\alpha)+B(k+\alpha k(k-1))]\,a_k z^k}\right|<1$$

Since Re $(z) < |z|$. We obtain after choosing the values of z on real axis and letting $z \to 1$
we get

$$\sum_{k=2}^{\infty}\{k+\alpha k^2-2\alpha k-(1-\alpha)-[B(k+\alpha k(k-1))-A(\alpha k+1-\alpha)]\}a_k \leq (A-B)$$

COROLLARY 1.1 If $f(z) \in \mathcal{H}(A,B,\alpha)$ then

$$a_k \leq \frac{(A-B)}{k+\alpha k^2-2\alpha k-(1-\alpha)-[B(k+\alpha k(k-1))-A(\alpha k+1-\alpha)]}$$

and the equality holds for

$$f(z)=z-\frac{(A-B)}{k+\alpha k^2-2\alpha k-(1-\alpha)-[B(k+\alpha k(k-1))-A(\alpha k+1-\alpha)]}z^k$$

THEOREM 1.2:A function $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \geq 0$ is in $K\mathcal{H}(A,B,\alpha)$ if and only if

$$\sum_{k=2}^{\infty}\{[k^2(1-2\alpha)+\alpha k^3-k(1-\alpha)]-A[-\alpha k^2-k(1-\alpha)]-B[k^2+\alpha k^2(k-1)]\}a_k \leq (A-B)$$

PROOF: Suppose $f(z)$ is in $K\mathcal{H}(A,B,\alpha)$

If and only if $zf'(z)$ is in $\mathcal{H}(A,B,\alpha)$
Let $g(z) = zf'(z)$

Therefore from(1.1) we have

$$\left|\frac{zg'(z)+\alpha z^2 g''(z)-\alpha zg'(z)-(1-\alpha)g(z)}{A[\alpha zg'(z)+(1-\alpha)g(z)]-B[zg'(z)+\alpha z^2 g''(z)]}\right|<1 \qquad \dots \dots (2.1)$$

$$\left|\frac{-\sum_{k=2}^{\infty}[k^2(1-2\alpha)+\alpha k^3-k(1-\alpha)]a_k z^k}{(A-B)z+\sum_{k=2}^{\infty}\{A[-\alpha k^2-k(1-\alpha)]+B[k^2+\alpha k^2(k-1)]\}a_k z^k}\right|<1$$

Since Re $(z) < |z|$. We obtain after choosing the values of z on real axis and letting $z \to 1$
we get

$$\sum_{k=2}^{\infty}\{[k^2(1-2\alpha)+\alpha k^3-k(1-\alpha)]-A[-\alpha k^2-k(1-\alpha)]-B[k^2+\alpha k^2(k-1)]\}a_k \le (A-B)$$

COROLLARY 1.2 If$f(z) \in K\mathcal{H}(A,B,\alpha)$ then

$$a_k \le \frac{(A-B)}{[k^2(1-2\alpha)+\alpha k^3-k(1-\alpha)]-A[-\alpha k^2-k(1-\alpha)]-B[k^2+\alpha k^2(k-1)]}$$

and the equality holds for

$$f(z) = z - \frac{(A-B)}{[k^2(1-2\alpha)+\alpha k^3-k(1-\alpha)]-A[-\alpha k^2-k(1-\alpha)]-B[k^2+\alpha k^2(k-1)]}z^k$$

Application of Fraction Calculus and Other Poperties

Various operators of fractional calculus have been studied in the literature rather extensively. Now we recall the following definitions.

DEFINITION 2.1 The integral operator studied by Bernardi is

$$L_c[f] = \frac{1+c}{z^c}\int_0^z f(x)x^{c-1}dx$$

DEFINITION 2.2 The Jung-Kim Srivastava operator is

$$I^{\sigma}f(z) = \frac{2^{\sigma}}{z\Gamma(\sigma)}\int_0^z \left(log\frac{z}{x}\right)^{\sigma-1} f(x)dx \quad ,\sigma > 0$$

$$= z - \sum_{k=2}^{\infty}\left(\frac{2}{1+k}\right)^{\sigma} a_k z^k$$

THEOREM 2.1: If $f \in \mathcal{H}(A,B,\alpha,\beta)$ then $L_c[f]$ is also in the class $\mathcal{H}(A,B,\alpha,\beta)$

PROOF: Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ then

$$L_c[f] = \frac{1+c}{z^c}\int_0^z \left(x - \sum_{k=2}^{\infty} a_k x^k\right) x^{c-1}dx$$

$$= \frac{1+c}{z^c}\left[\left(\frac{1}{c+1}x^{c+1} - \sum_{k=2}^{\infty}\frac{1}{k+c}a_k x^{k+c}\right)_0^z\right]$$

$$= z - \sum_{k=2}^{\infty}\frac{1+c}{k+c}a_k z^k$$

Since $c > -1$, $k \ge 2$ then $\frac{1+c}{k+c} \le 1$ so we have

$$\sum_{k=2}^{\infty}\frac{\{k+\alpha k^2-2\alpha k-(1-\alpha)-[B(k+\alpha k(k-1))-A(\alpha k+1-\alpha)]\}}{(A-B)}\left(\frac{1+c}{k+c}\right)a_k$$

$$\le \sum_{k=2}^{\infty}\frac{\{k+\alpha k^2-2\alpha k-(1-\alpha)-[B(k+\alpha k(k-1))-A(\alpha k+1-\alpha)]\}}{(A-B)}a_k < 1$$

Therefore $L_c[f]$ is also in the class $\mathcal{H}(A,B,\alpha,\beta)$
Similarly we can prove if $f \in K\mathcal{H}(A,B,\alpha,\beta)$ then $L_c[f]$ is also in the class $K\mathcal{H}(A,B,\alpha,\beta)$

THEOREM 2.2 : Let $c \in \mathbb{R}, c > -1$. If $L_c[f]$ is in the class $\mathcal{H}(A,B,\alpha,\beta)$ then $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$is univalent in $|z| < r$ when

$$r = \inf_k \left(\left(\frac{(k+c)\{k+\alpha k^2 - 2\alpha k - (1-\alpha) - [B(k+\alpha k(k-1)) - A(\alpha k + 1 - \alpha)]\}}{k(1+c)(A-B)} \right)^{\frac{1}{k-1}} \right)$$

PROOF: Let

$$L_c[f] = z - \sum_{k=2}^{\infty} d_k z^k = \frac{1+c}{z^c} \int_0^z f(x) x^{c-1} dx$$

So

$$f(z) = z - \sum_{k=2}^{\infty} \frac{1+c}{k+c} d_k z^k \quad , c > -1$$

Therefore it is sufficient to prove that

$$|f'(z) - 1| < 1$$

$$\big||f'(z) - 1|\big| = \left| -\sum_{k=2}^{\infty} \frac{k(1+c)}{k+c} d_k z^{k-1} \right|$$

Therefore

$$\sum_{k=2}^{\infty} \frac{k(1+c)}{k+c} d_k |z|^{k-1} \leq 1$$

Also since $L_c[f]$ is in the class $\mathcal{H}(A,B,\alpha,\beta)$
Therefore

$$\sum_{k=2}^{\infty} \frac{\{k+\alpha k^2 - 2\alpha k - (1-\alpha) - [B(k+\alpha k(k-1)) - A(\alpha k + 1 - \alpha)]\}}{(A-B)} d_k \leq 1$$

$$\frac{k(1+c)}{k+c} |z|^{k-1} \leq \frac{\{k+\alpha k^2 - 2\alpha k - (1-\alpha) - [B(k+\alpha k(k-1)) - A(\alpha k + 1 - \alpha)]\}}{(A-B)}$$

$$|z| \leq \left[\frac{(k+c)\{k+\alpha k^2 - 2\alpha k - (1-\alpha) - [B(k+\alpha k(k-1)) - A(\alpha k + 1 - \alpha)]\}}{k(1+c)(A-B)} \right]^{\frac{1}{k-1}}$$

Hence the proof of theorem is complete.

THEOREM 2.3 Let $c \in \mathbb{R}, c > -1$. If $L_c[f]$ is in the class $K\mathcal{H}(A,B,\alpha,\beta)$ then $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$is univalent in $|z| < r$ when

$$r = \inf_k \left(\left(\frac{(k+c)\{k+\alpha k^2 - 2\alpha k - (1-\alpha) - [B(k+\alpha k(k-1)) - A(\alpha k + 1 - \alpha)]\}}{(1+c)(A-B)} \right)^{\frac{1}{k-1}} \right)$$

THEOREM 2.4 Let $f \in \mathcal{H}(A,B,\alpha,\beta)$. Then the Jung-Kim Srivastava operator is

$$I^\sigma f(z) = \frac{2^\sigma}{z\Gamma(\sigma)} \int_0^z \left(\log \frac{z}{x} \right)^{\sigma-1} f(x) dx \quad , \sigma > 0$$

is also in the class $\mathcal{H}(A,B,\alpha,\beta)$

PROOF: We have

$$I^\sigma f(z) = z - \sum_{k=2}^{\infty} \left(\frac{2}{1+k} \right)^\sigma a_k z^k$$

Since

$$\left(\frac{2}{1+k} \right)^\sigma \leq 1 \; , \quad \sigma > 0$$

We have

$$\sum_{k=2}^{\infty} \{k+\alpha k^2 - 2\alpha k - (1-\alpha) - [B(k+\alpha k(k-1)) - A(\alpha k + 1 - \alpha)]\} \left(\frac{2}{1+k} \right)^\sigma a_k$$

$$\leq \sum_{k=2}^{\infty}\{k+\alpha k^2-2\alpha k-(1-\alpha)-[B(k+\alpha k(k-1))-A(\alpha k+1-\alpha)]\}a_k \leq (A-B)$$

Therefore, by THEOREM 1.1 we have

$I^{\sigma} f(z) \in \mathcal{H}(A,B,\alpha,\beta)$

THEOREM 2.5: Let $\in K\mathcal{H}(A,B,\alpha,\beta)$. Then the Jung-Kim Srivastava operator is

$$I^{\sigma} f(z) = \frac{2^{\sigma}}{z\Gamma(\sigma)}\int_0^z \left(log\frac{z}{x}\right)^{\sigma-1} f(x)dx \quad ,\sigma > 0$$

is also in the class $K\mathcal{H}(A,B,\alpha,\beta)$

THEOREM 2.6: Let $f \in \mathcal{H}(A,B,\alpha,\beta)$ then for every $\chi \geq 0$ then the function

$$L_{\chi}(z) = (1-\chi)f(z) + \chi\int_0^z \frac{f(y)}{y}\, dy$$

is also in the class $\mathcal{H}(A,B,\alpha,\beta)$

PROOF

$$L_{\chi}(z) = (1-\chi)\left(z-\sum_{k=2}^{\infty} a_k z^k\right) + \chi\int_0^z \frac{y-\sum_{k=2}^{\infty} a_k y^k}{y}\, dy$$

$$= z-\chi z-\sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{\infty}\chi a_k z^k + \chi z - \sum_{k=2}^{\infty} a_k \frac{\chi}{k} z^k$$

$$= z-\sum_{k=2}^{\infty}\left(1-\chi+\frac{\chi}{k}\right) a_k z^k$$

Since $\left(1-\chi+\frac{\chi}{k}\right) < 1$

Therefore we have

$$\sum_{k=2}^{\infty}\frac{\{k+\alpha k^2-2\alpha k-(1-\alpha)-[B(k+\alpha k(k-1))-A(\alpha k+1-\alpha)]\}\left(1-\chi+\frac{\chi}{k}\right)}{(A-B)} a_k \leq 1$$

by THEOREM 1.1 we have

$L_{\chi}(z) \in \mathcal{H}(A,B,\alpha,\beta)$

THEOREM 2.7: Let $f \in K\mathcal{H}(A,B,\alpha,\beta)$ then for every $\chi \geq 0$ then the function

$$L_{\chi}(z) = (1-\chi)f(z) + \chi\int_0^z \frac{f(y)}{y}\, dy$$

is also in the class $K\mathcal{H}(A,B,\alpha,\beta)$

THEOREM 2.8: Let $f \in \mathcal{H}(A,B,\alpha,\beta)$ then for every $\chi \geq 0$ then the function

$$M_{\chi}(z) = (1-\chi)z + \chi\int_0^z \frac{f(y)}{y}\, dy$$

is also in the class $\mathcal{H}(A,B,\alpha,\beta)$

Proof

$$H_{\chi}(z) = (1-\chi)z + \chi\int_0^z \frac{y-\sum_{k=2}^{\infty} a_k y^k}{y}\, dy$$

$$= z-\chi z + \chi z - \sum_{k=2}^{\infty} a_k \frac{\chi}{k} z^k$$

$$= z - \sum_{k=2}^{\infty} a_k \frac{\chi}{k} z^k$$

Since $\frac{\chi}{k} < 1$

Therefore we have

$$\sum_{k=2}^{\infty} \frac{\{k + \alpha k^2 - 2\alpha k - (1-\alpha) - [B(k + \alpha k(k-1)) - A(\alpha k + 1 - \alpha)]\}\left(\frac{\chi}{k}\right)}{(A-B)} a_k \leq 1$$

by THEOREM 1.1 we have

$M_\chi(z) \in \mathcal{H}(A, B, \alpha, \beta)$

THEOREM 2.9: Let $f \in K\mathcal{H}(A, B, \alpha, \beta)$ then for every $\chi \geq 0$ then the function

$$M_\chi(z) = (1-\chi)z + \chi \int_0^z \frac{f(y)}{y}\, dy$$

is also in the class $K\mathcal{H}(A, B, \alpha, \beta)$

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