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SOME NOTES ABOUT CI-REGULAR LOCAL RINGS

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ABSTRACT

Let (R, m, k) be a commutative Noetherian local ring. In this paper, we define CI-regular local rings and we will study some of their properties. Also, we will characterize regular local rings.

INTRODUCTION

Prerequisites

Throughout this paper, (R, m) is a local ring and all rings are commutative and Noetherian with identity. The projective dimension is a familiar and famous numerical invariant in the classical homological algebra. One of the fascinating theorems which is related to this dimension, is Auslander-Buchsbaum-Serre Theorem ((Auslander *et al*, 1956) and (Serre, 1956)) which asserts thatR is a regular ring if every finitely generated R-modules has finite projective dimension. Motivated by this, Auslander and Bridger (Auslander *et al*, 1969), introduced the Gorenstein dimension (abbr. G-dimension) for any finitely generated R-modules and they proved that R is Gorenstein when every finitely generated R-module has finite G-dimension.

The G-dimension has a very essential role for studying Gorenstein homological algebra and it was studied in more details in (Auslander *et al*, 1969) and (Enochs *et al*, 2000). Let us recall the definition of G-dimension. Let M be a nonzero finitely generated R-module. The G-dimension of M is zero, $Gdim_R M = 0$, if and only if the natural homomorphism $M \to Hom_R(Hom_R(M,R),R)$ is an isomorphism and $Ext^i_R(M,R) = 0 = Ext^i_R(Hom_R(M,R),R)$ for any $i > 0$. We set $Gdim_R 0 = -\infty$. Also, for an integer n, $Gdim_R M \leq$ n if and only if there is an exact sequence
$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$
of R-modules such that $Gdim_R X_i = 0$ for any $0 \leq i \leq n$.
More recently, Avramov, Gasharov and Peeva (Avramov *et al*, 1997) introduced the concept of complete intersection dimension for finitely generated R-modules as a generalization of projective dimension. They proved that R is complete intersection when every finitely generated R-module has finite complete intersection dimension. For defining complete intersection dimension, we need the definition of quasi-deformation of R. A quasi-deformation of R is a diagram of local ring homomorphisms$R \to R' \leftarrow Q$ such that $R \to R'$ is faithfully flat and $R' \leftarrow Q$ is surjective with the kernel which is generated by a Q-regular sequence.

The complete intersection dimension of a finitely generated R-module M,$CIdim_R M$ is defined as follow:

$$CIdim_R M = \inf\{pd_Q(M \otimes_R R') - pd_Q R' \mid R \to R' \leftarrow Q \text{ is a quasi-deformation}\}.$$

These homological dimensions satisfy in the following inequalities

$$Gdim_R M \leq CIdim_R M \leq pd_R M \qquad (1)$$

with equality to the left of any finite quantity, see (Avramov *et al*, 1997).

We denote the category of finitely generated R-modules by mod(R), the subcategory of mod(R) consisting of all free R-modules by F(R), the subcategory of mod(R) consisting of zero module and all R-modules M such that $Gdim_R M = 0$ (resp.

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$CIdim_R M = 0$) by G(R) (resp. CI(R)). By using (1), we have the following inclusion relations between the subcategories of mod(R) F R □ CI R □ G R .

Takahashi (Takahashi, 2008) defined R to be G-regular if G(R)=F(R). We define R to be CI-regular if CI(R)=F(R). In this paper, we study some properties for CI-regular local rings and we will show that if □^{m} R is a CI-regular ring, then R is a CI-regular ring too. Also, we will prove that R is regular if and only if R is complete intersection and CI-regular.

RESULTS

We start this section by the following definition.

Definition 1. We say that a local ring (R, m) is CI-regular if CI(R) coincides with F(R).

Lemma 2. Let (R, m) be a local ring and M be a finitely generated R-module.

For a faithfully flat local ring homomorphism $R \to S$, there is an inequality

$CIdim_R M \leq CIdim_S(M \otimes_R S)$

with equality when $CIdim_S(M \otimes_R S)$ is finite.

Let $f: Q \to R$ be a surjective local ring homomorphism with kernel generated by Q-regular sequence $x = x_1, \ldots, x_n$. Then

$CIdim_R M \leq CIdim_Q M - n$

When xM=0 and also if $x = x_1, \ldots, x_n$ is an M-regular sequence, then

$CIdim_R \frac{M}{xM} \leq CIdim_Q M,$

with equality when $CIdim_Q M$ is finite.

Let a be an ideal of R and □^{a} R be the a-adic completion of R. Then
$CIdim_R M = CIdim_{□^a R}(M \otimes_R □^a R)$.

If $CIdim_R M$ is finite, then $CIdim_R M = dept□ R - dept□_R M$.

Proof. See (Avramov *et al*, 1997).

Lemma 3. Let (R, m) be a local ring and □^{m} R the m-adic completion of R. If □^{m} R is a CI-regular ring, then R is a CI-regular ring too.

Proof. Let M be a finitely generated R-module such that $CIdim_R M = 0$. Then $M \otimes_R$ □^{m} R is a finitely generated □^{m} R -module and $CIdim_R M = CIdim_{□^m R}(M \otimes_R □^m R)$ by Lemma 2(3). Since □^{m} R is CI-regular ring, then $M \otimes_R$ □^{m} R is a free □^{m} R -module. Then M is a free R-module by (Avramov, 1965).

Lemma 4. Let (R, m) be a local ring and $x = x_1, \ldots, x_n$ an R-sequence. If $\frac{R}{xR}$ is a CI-regular ring, then R is CI-regular ring, too.

Proof. By induction it is enough to consider the case in which x consists of a single R-regular element x. Let M be a finitely generated R-module such that $CIdim_R M = 0$. Since $Gdim_R M = 0$, then x is an M-regular element by (Christensen , 2010). Then $CIdim_{\frac{R}{xR}} \frac{M}{xM} = 0$by Lemma 2(2) and so $\frac{M}{xM}$ is a free $\frac{R}{xR}$-module by the assumption. So M is a free R-module by (Bruns *et al*, 1993).

Note that the converse of the above Lemma is not necessary correct in general.

Example 5. Let R □= $K[[x]]$, where K is a field. So R is a regular ring and then it is CI-regular. But $S := \frac{R}{x^2 R}$ is not CI-regular ring. Because $CIdim_S K$ is finite by (Avramov *et al*, 1997) while $pd_S K$ is infinite.

Remark 6. Let $f: R \to S$ be a local ring homomorphism and R is a CI-regular. If $fd_R f$ is finite, then may be S is not CI-regular ring. Assume that R □= $K[[x]]$, where K is a field and $S := \frac{R}{x^2 R}$. Let $f: R \to S$ be the natural local ring homomorphism. Then Example 5 shows that S is not CI-regular while R is a CI-regular ring.

Proposition 7. Let (R, m) be a local ring. If R is CI-regular ring, then the formal power series ring R[[$X_1, \ldots, X_n$]] is CI-regular ring.

Proof. Since R □ $\frac{R[[X_1,\ldots,X_n]]}{(X_1,\ldots,X_n)}$, the assertion follows from Lemma 4.

Lemma 8. Let (R, m) be a local ring. Then R is CI-regular if and only if $CIdim_R M = pd_R M$.

Proof.(□) Let M be a finitely generated R-module. It suffices to show that $CIdim_R M \geq pd_R M$.Without loss of generality, we assume that $CIdim_R M$ is finite. Set $n := CIdim_R M$. By (Wagstaff, 2004), there exists an exact sequence

$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0,$

such that$CIdim_R X_i = 0$ for any $0 \leq i \leq n$ and so by the assumption,X_i is projective for any $0 \leq i \leq n$ which implies that $pd_R M \leq n$.
(□)is trivial.

Next, we present a criterion for specification regular local rings.

Theorem 9. Let (R, m, k) be a local ring. The following are equivalent:

R is regular,

R is complete intersection and CI-regular.

Proof.(i □ ii) By (Bruns *et al*, 1993), R is complete intersection and for any finitely generated R-module M, $pd_R M$ is finite, and so by (Avramov *et al*, 1997), $pd_R M = CIdimd_R M$ which implies that R is CI-regular by Lemma 8.

ii □ i By (Avramov *et al*, 1997) and Lemma 8, $pd_R k$ is finite, and so R is regular by (Bruns *et al*, 1993).

Definition 10. Let (R, m) be a local ring. The global CI-dimension of R is

$$glCI - dim(R) \text{ □= } \sup\{CI - dim_R M \mid M \text{ is finitely generated } R - module\}$$

Lemma11. Let (R, m, k) be a Noetherian local ring. If $glCI - dim(R)$ is finite, then $glCI - dim\ R = CIdim_R k$.

Proof. It is clear that $CIdim_R k \leq glCI - \dim(R)$. Since $glCI - dim(R)$ is finite, then $CIdim_R M$ is finite for any finitely generated R-modules M. So $CIdim_R M \leq depth R$ by Lemma 2 (4). As $\mathrm{depth}\, R = \mathrm{CIdim}_R k$, the assertion is followed.

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