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Research Article

ON SECOND HANKEL DETERMINANT FOR STARLIKE AND CONVEX FUNCTIONS

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ABSTRACT

A denote the class of functions which are analytic, normalized and univalent in the open disc $D = \{z: |z| < 1\}$. The important subclasses of A are starlike and convex functions which are denoted by $S^*(\gamma)$ and $C(\gamma)$. This paper focuses on attaining sharp upper bounds for the functional $|a_2a_4 - a_3{}^2|$ for univalent functions.

INTRODUCTION

Let A denote the class of normalized, analytic and univalent function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad \text{where } z \in D = \{z: |z| < 1\} \tag{1.1}$$

The q^{th} Hankel determinant for $q \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & \cdots & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ a_{n+q-1} & \cdots & \cdots & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}$$

This determinant also been considered by several authors. In particular, sharp bounds on $H_2(2)$ were obtained by the authors of articles [2], [3], [4], [8] for different classes.

In particular, $q = 1, n = 1, a_1 = 1$ and $q = 2, n = 2$ the Hankel determinant simplifies respectively to $H_2(1)$ and $H_2(2)$, both are second Hankel determinant. Here $H_2(1)$ also called as Fekete and Szego functional.

These are simplifies respectively as $H_2(1) = |a_3 - a_2{}^2|$, $H_2(2) = |a_2a_4 - a_3{}^2|$.

In this paper we consider $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$ that is $H_2(2) = |a_2a_4 - a_3{}^2|$. We have to obtain upper bound for the functional $|a_2a_4 - a_3{}^2|$ for functions belonging to the following classes.

Definition (1.1)

A function $f(z) \in A$ is said to be starlike of complex order γ $(\gamma \in \mathbb{C}/\{0\})$, that is $f \in S^*(\gamma)$ if it satisfies the inequality

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$$Re\left\{1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0 \qquad (z\in D,\gamma\in\mathbb{C}/\{0\}) \tag{1.2}$$

The choice of $\gamma = 1$ yields $Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$, $z\in D$ the class of starlike functions S^*.

Definition (1.2)

A function $f(z)\in A$ is said to be convex of complex order γ $(\gamma\in\mathbb{C}/\{0\})$, that is $f\in C(\gamma)$ if it satisfies the inequality

$$Re\left\{1+\frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\}>0 \qquad (z\in D,\gamma\in\mathbb{C}/\{0\}) \tag{1.3}$$

The choice of $\gamma = 1$ yields $Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0$, $z\in D$ the class of convex functions C.

Preliminary Results

Let M be the family of all functions p analytic in D for which $Re\{p(z)\} > 0$ and

$$p(z) = 1+c_1z+c_2z^2+\ldots\ldots.. \tag{2.1}$$

For $z\in D$.
Lemma (2.1): [5] If $p\in M$ then $|c_k|\le 2$ for each $k\in N$.

Lemma (2.2): ([6], [7]) Let $p\in M$ then $2c_2 = c_1{}^2 + x(4-c_1{}^2)$ (2.2)

And

$$4c_3 = c_1{}^3+2(4-c_1{}^2)c_1x-c_1(4-c_1{}^2)x^2+2(4-c_1{}^2)(1-|x|^2)z \tag{2.3}$$

For some values of x, z such that, $|x|\le 1$ & $|z|\le 1$.
Theorem (2.1): [1] Let $f\in S^*$. Then $|a_2a_4-a_3{}^2|\le 1$.
The result obtained is sharp.
Theorem (2.1): [1] Let $f\in C$. Then $|a_2a_4-a_3{}^2|\le \frac{1}{8}$.
The result obtained is sharp.

Main Results

Theorem (3.1): Let $f\in S^*(\gamma)$. Then $|a_2a_4-a_3{}^2|\le \frac{8\gamma^2}{(3-\gamma)^2}$
The result obtained is sharp.

Proof: $f\in S^*(\gamma)$ then $\exists\, p\in M$ such that $zf'(z) = \gamma f(z)p(z)$ (3.1)

for some $z\in D$. Equating the coefficients in (3.1) yields

$$a_2 = \frac{\gamma c_1}{2-\gamma}$$
$$a_3 = \frac{\gamma c_2}{3-\gamma}+\frac{\gamma^2c_1{}^2}{(2-\gamma)(3-\gamma)}$$
$$a_4 = \frac{\gamma c_3}{4-\gamma}+\frac{\gamma^2c_1c_2(5-2\gamma)}{(2-\gamma)(3-\gamma)(4-\gamma)}+\frac{\gamma^2c_1{}^3}{(2-\gamma)(3-\gamma)(4-\gamma)} \tag{3.2}$$

From (3.2), it is easily established that,

$$|a_2a_4-a_3{}^2| = \left|\frac{\gamma c_1c_3}{(2-\gamma)(4-\gamma)}-\frac{\gamma^2c_2{}^2}{(3-\gamma)^2}-\frac{\gamma^3c_1{}^2\left(c_1{}^2(4+2\gamma-\gamma^2)+c_2(4\gamma^2-23\gamma+31)\right)}{(2-\gamma)^2(3-\gamma)^2(4-\gamma)}\right|$$

Substituting for c_2 and c_3 from (2.1) and (2.2) and since $|c_1|\le 2$ by lemma (2.1), $c_1 = c$ and assume without restriction $c\in[0,2]$. We obtain,

$$|a_2a_4-a_3{}^2| = \left|\frac{\gamma\left(c^4+2c^2(4-c^2)x-c^2(4-c^2)x^2+2c(4-c^2)(1-|x|^2)z\right)}{4(2-\gamma)(4-\gamma)}-\frac{\gamma^2\left(c^2+x(4-c^2)\right)^2}{2(3-\gamma)^2}\right.$$
$$\left.-\frac{\gamma^3c^4(4+2\gamma-\gamma^2)}{(2-\gamma)^2(3-\gamma)^2(4-\gamma)}-\frac{\gamma^3c^2\left(c^4+c^2x(4-c^2)\right)(4\gamma^2-23\gamma+31)}{2(2-\gamma)^2(3-\gamma)^2(4-\gamma)}\right|$$

Using triangle inequality,

$$|a_2a_4 - a_3^2| \le \frac{\gamma\left(c^4 + 2c^2(4-c^2)\rho + 2c(4-c^2) + c(4-c^2)(c-2)\rho^2\right)}{4(2-\gamma)(4-\gamma)} + \frac{\gamma^2(c^4 + (4-c^2)^2\rho^2 + 2c^2(4-c^2)\rho)}{2(3-\gamma)^2}$$
$$+ \frac{\gamma^3c^4(4+2\gamma-\gamma^2)}{(2-\gamma)^2(3-\gamma)^2(4-\gamma)} + \frac{\gamma^3c^2\left(c^4 + c^2(4-c^2)\rho\right)(4\gamma^2 - 23\gamma + 31)}{2(2-\gamma)^2(3-\gamma)^2(4-\gamma)}$$

$$|a_2a_4 - a_3^2| = F(\rho) \qquad (3.4)$$

With $\rho = |x| \le 1$. Furthermore,

$$F'(\rho) = \frac{\gamma\left(c^2(4-c^2) + c(4-c^2)(c-2)\rho\right)}{2(2-\gamma)(4-\gamma)} + \frac{\gamma^2((4-c^2)^2\rho + c^2(4-c^2))}{(3-\gamma)} + \frac{\gamma^3c^2(c^2(4-c^2))(4\gamma^2 - 23\gamma + 31)}{2(2-\gamma)^2(3-\gamma)^2(4-\gamma)}$$

and with elementary calculus , we can show that $F'(\rho) > 0$ for $\rho > 0$, implying that F is an increasing function and thus the upper bound for (3.4) correspond to $\rho = 1$ and $c = 0$ gives,

$|a_2a_4 - a_3^2| \le \frac{8\gamma^2}{(3-\gamma)^2}$. This completes the proof.

For $\gamma = 0$, $|a_2a_4 - a_3^2| \le 0$ which is ≤ 1, the result sharp obtained by Aini Janteng [1].

Theorem (3.2): Let $f \in C(\gamma)$. Then $|a_2a_4 - a_3^2| \le \frac{\gamma^2}{9}$

The result is obtained sharp.

Proof: $f \in C(\gamma)$ then $\exists\, p \in M$ such that

$$\gamma f'(z) + zf''(z) = \gamma f'(z)\,p(z) \qquad (3.5)$$

for some $z \in D$. Equating the coefficients in (3.5) yields

$$a_2 = \frac{c_1\gamma}{2}$$
$$a_3 = \frac{\gamma(c_1^2 + c_2)}{6}$$
$$a_4 = \frac{\gamma(2c_3 + 2c_1c_2\gamma + c_1(c_1^2 + c_2)\gamma^2)}{24} \qquad (3.6)$$

From (3.6), it is easily established that,

$$|a_2a_4 - a_3^2| = \frac{c_1\gamma^2(2c_3 + c_1^3\gamma^2 + \gamma c_1c_2(2+\gamma))}{48} - \frac{\gamma^2(c_1^4 + 2c_1c_2 + c_2^2)}{36}$$

Substituting for c_2 and c_3 from (2.1) and (2.2) and since $|c_1| \le 2$ by lemma (2.1), $c_1 = c$ and assume without restriction $c \in [0,2]$. We obtain,

$$|a_2a_4 - a_3^2| = \frac{\gamma^2c\left(c^3 + 2(4-c^2)cx - c(4-c^2)x^2 + 2(4-c^2)(1-|x|^2)z\right)}{96} + \frac{\gamma^2c^4(3\gamma^2-4)}{144} + \frac{\gamma^2(c^2 + x(4-c^2))^2}{144}$$
$$+ \frac{\gamma^2c\left(3c(2+\gamma) - 8\right)(c^2 + x(4-c^2))}{288}$$

Using triangle inequality,

$$|a_2a_4 - a_3^2| \le \frac{\gamma^2\left(c^4 + 2c^2(4-c^2)\rho + 2c(4-c^2) + c(4-c^2)(c-2)\rho^2\right)}{96} + \frac{\gamma^2c^4(3\gamma^2-4)}{144} + \frac{\gamma^2\left(c^4 + 2c^2(4-c^2)\rho + (4-c^2)^2\rho^2\right)}{144} + \frac{\gamma^2\left(3c^2(2+\gamma) - 8c\right)(c^2 + (4-c^2)\rho)}{288}$$
(3.7)

$|a_2a_4 - a_3^2| = F(\rho)$ with $\rho = |x| \le 1$

Furthermore $F'(\rho) = \frac{\gamma^2\left(c^2(4-c^2) + c(4-c^2)(c-2)\rho\right)}{48} + \frac{\gamma^2(c^2(4-c^2) + (4-c^2)^2\rho)}{72} + \frac{\gamma^2\left(3c^2(2+\gamma) - 8c\right)(4-c^2)}{288}$

And with elementary calculus, we can show that $F'(\rho) > 0$ for $\rho > 0$, implying that F is an increasing function and thus the upper bound for (3.6) correspond to $\rho = 1$ and $c = 0$ gives,

$$|a_2a_4 - a_3^2| \le \frac{\gamma^2}{9}$$

For $\gamma = 1$, $|a_2a_4 - a_3^2| \le \frac{1}{9}$ which is $\le \frac{1}{8}$, the result sharp obtained by Aini Janteng [1].

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