ISSN: 0976-3031

Available Online at http://www.recentscientific.com

International Journal of Recent Scientific Research
Vol. 7, Issue, 11, pp. 14356-14358, November, 2016

International Journal of Recent Scientific Research

Research Article

GENERALISED SECOND HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS

Harjinder Singh

O/o Director Public Instructions (Colleges), Punjab, Chandigarh

ARTICLE INFO

Article History:

Received 17th August, 2016
Received in revised form 21th September, 2016
Accepted 28th October, 2016
Published online 28th November, 2016

Key Words:

Hankel determinant, Carathéodory functions, Univalent Starlike, Univalent convex, close-to-convex and close-to-starlike functions with respect to symmetric points.

ABSTRACT

The purpose of the present paper is to consider some subclasses of generalized Pascu classes of functions with respect to symmetric points and obtain sharp upper bounds for the generalized second Hankel determinant $|a_2a_4 - \mu a_3^2|$ for an analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ μ is real and $|z| < 1$ belonging to these classes.

INTRODUCTION

Definitions

Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in $E = \{z: |z| < 1\}$ and S is the subclass of functions in A which are univalent.

The functions of the form $P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ which are regular in the open unit disc $E = \{z: |z| < 1\}$ with the condition $\mathrm{Re}\, P(z) > 0$ are **Carathéodory Functions** [1]. Let $\wp$ denote the class of Carathéodory Functions.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in E. For $q \geq 1$, the q^{th} Hankel determinant [7, 8]of f is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

We are interested in the particular case when $q = n = 2$. The second Hankel determinant was studied by various authors including Hayman [5] and Pommeranke[9, 10]. We are interested in sharp upper bounds for the functional $|a_2a_4 - \mu a_3^2|$ for certain subclasses of analytic functions.

Sakaguchi [11] introduced the concept of univalent starlike functions with respect to symmetric points. A function $f \in A$ is univalent starlike with respect to symmetric points if and only if $Re\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0$,and the class of functions may be denoted by S_s^*.

Das and Singh [2] extended the concept of symmetric points to convex and close-to-convex functions. A function $f \in A$ is said to be univalent convex w. r. t. symmetric points if and only if $Re\left(\frac{(zf'(z))'}{(f(z)-f(-z))'}\right) > 0$ and class of such functions is denoted by K_s.

C_s is the class of close-to-convex functions f in A with respect to symmetric points if there exists a function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S_s^*$ such that $Re\left(\frac{zf'(z)}{g(z)-g(-z)}\right) > 0$.

If there exists a function $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in K_s$ for which $Re\left(\frac{(zf'(z))'}{(h(z)-h(-z))'}\right) > 0$,

then the class of functions $f(z)$ in A may be denoted by $C_{1(s)}$.

Let C_s^* denote the class of functions in A which satisfy the condition $Re\left(\frac{f(z)}{g(z)-g(-z)}\right) > 0$, $g \in S_s^*$. On replacing g by $h \in K_s$, the corresponding class may be denoted by $C_{1(s)}^*$.

*Corresponding author: ***Harjinder Singh***
O/o Director Public Instructions (Colleges), Punjab, Chandigarh

Let $\alpha \geq 0$ and $\frac{f(z)f'(z)}{z} \neq 0$. Then $C_s^*(\alpha)$ is the class of functions f in A with respect to symmetric points if there exists a function $g \in S_s^*$ such that $\frac{2(1-\alpha)f(z)}{g(z)-g(-z)} + \frac{2\alpha z f'(z)}{g(z)-g(-z)} = P(z)$. If g is replaced by $h \in K_s$ in the condition of $C_s^*(\alpha)$, the corresponding class is denoted by $C_{1(s)}^*(\alpha)$. Recently Singh and Singh [4] obtained obtained the estimates of second Hankel determinant for the classes $C_s^*(\alpha)$ and $C_{1(s)}^*(\alpha)$

The classes $T_s^*(\alpha)$ and $T_{1(s)}^*(\alpha)$ are respectively defined as

$$f \in A;\ \frac{2zf'(z)}{g(z)-g(-z)} + \frac{2\alpha z^2 f''(z)}{g(z)-g(-z)} = P(z),\ g \in S_s^* \text{ and}$$

$$f \in A;\ \frac{2zf'(z)}{h(z)-h(-z)} + \frac{2\alpha z^2 f''(z)}{h(z)-h(-z)} = P(z),\ h \in K_s.$$

Preliminary Lemmas

The following lemmas are required to establish our results.

Lemma 2.1 ([3]). If $P(z) \in \mathcal{P}$, then $|p_k| \leq 2\ (k = 1,2,3,\ldots)$.
Lemma 2.2 ([6]). If $P(z) \in \mathcal{P}$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x, \tag{2.1}$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z, \tag{2.2}$$

for some x and z with $|x| \leq 1$ and $|z| \leq 1$.

MAIN RESULTS

THEOREM 3.3 Let $f \in T_s^*(\alpha)$. Then

$$|a_2a_4 - \mu a_3^2| \leq \begin{cases} \frac{8(2B-\mu K)^2}{C(B-\mu K)} - \frac{\mu}{(1+2\alpha)^2} & \text{if } \mu \leq 0; & (3.1) \\ \frac{32}{C}(B-\mu K) + \frac{\mu}{(1+2\alpha)^2} & \text{if } 0 \leq \mu \leq \frac{B}{K}; & (3.2) \\ \frac{\mu}{(1+2\alpha)^2} & \text{if } \frac{B}{K} \leq \mu \leq \frac{2B}{K}; & (3.3) \\ \frac{8(\mu K - 2B)^2}{C(\mu K - B)} + \frac{\mu}{(1+2\alpha)^2} & \text{if } \mu \geq \frac{2B}{K}, & (3.4) \end{cases}$$

where

$$B = (1+2\alpha)^2$$
$$K = 2(1+\alpha)(1+3\alpha) \tag{3.5}$$
$$C = 32(1+\alpha)(1+3\alpha)(1+2\alpha)^2$$

The results are sharp.

Proof. Since $f \in T_s^*(\alpha)$, we have

$$\frac{2zf'(z)}{g(z)-g(-z)} + \frac{2\alpha z^2 f''(z)}{g(z)-g(-z)} = P(z). \tag{3.6}$$

Equating the coefficients in (3.6), it is easily established that

$$a_2 = \frac{p_1}{2(1+\alpha)}$$
$$a_3 = \frac{p_2 + b_3}{3(1+2\alpha)} \tag{3.7}$$
$$a_4 = \frac{p_3 + p_1b_3}{4(1+3\alpha)}$$

Again $g \in S_s^*$ implies that

$$\frac{2zg'(z)}{g(z)-g(-z)} = P(z). \tag{3.8}$$

Identifying the terms in (3.8), we get

$$b_2 = \frac{p_1}{2}$$
$$b_3 = \frac{p_2}{2} \tag{3.9}$$
$$b_4 = \frac{p_3}{4} + \frac{p_1p_2}{8}$$

Combination of (3.7) and (3.9) gives

$$a_2 = \frac{p_1}{2(1+\alpha)}$$
$$a_3 = \frac{p_2}{2(1+2\alpha)} \tag{3.10}$$
$$a_4 = \frac{2p_3 + p_1p_2}{8(1+3\alpha)}$$

System (3.10) and (3.5) ensures that

$$C(a_2a_4 - \mu a_3^2) = Bp_1(4p_3) + Bp_1^2(2p_2) - \mu K(2p_2)^2$$

which, by lemma (2.2), can be written as

$$C(a_2a_4 - \mu a_3^2) = Bp_1\left(p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z\right)$$
$$+ Bp_1^2\left(p_1^2 + (4 - p_1^2)x\right) - \mu K\left(p_1^2 + (4 - p_1^2)x\right)^2$$

for some x and z with $|x| \leq 1$ and $|z| \leq 1$.

$$C(a_2a_4 - \mu a_3^2) = (2B - \mu K)p_1^4 + (3B - 2\mu K)p_1^2(4 - p_1^2)x$$
$$- (4 - p_1^2)\left((B - \mu K)p_1^2 + 4\mu K\right)x^2 + 2Bp_1(4 - p_1^2)(1 - x^2)z. \tag{3.11}$$

Replacing p_1 by $p \in [0,2]$ and using triangular inequality, (3.11) takes the form

$$C|a_2a_4 - \mu a_3^2| = |2B - \mu K|p^4 + |3B - 2\mu K|p^2(4 - p^2)\delta + 2Bp(4 - p^2)(1 - \delta^2)$$
$$+ (4 - p^2)\left(|B - \mu K|p^2 + 4|\mu|K\right)\delta^2,\ (\delta = |x| \leq 1),$$

which can be put in the form

$$C|a_2a_4 - \mu a_3^2| \leq \tag{3.12}$$

$$\begin{cases} (2B - \mu K)p^4 + (3B - 2\mu K)p^2(4 - p^2)\delta + 2Bp(4 - p^2) - (4 - p^2)\left((B - \mu K)p^2 - 4\mu K - 2Bp\right)\delta^2 & \text{if } \mu \leq 0; \\ (2B - \mu K)p^4 + (3B - 2\mu K)p^2(4 - p^2)\delta + 2Bp(4 - p^2) + (4 - p^2)\left((B - \mu K)p^2 + 4\mu K - 2Bp\right)\delta^2 & \text{if } 0 \leq \mu \leq \frac{B}{K}; \\ (2B - \mu K)p^4 + (3B - 2\mu K)p^2(4 - p^2)\delta + 2Bp(4 - p^2) + (4 - p^2)\left((\mu K - B)p^2 + 4\mu K - 2Bp\right)\delta^2 & \text{if } \frac{B}{K} \leq \mu \leq \frac{3B}{2K}; \\ (2B - \mu K)p^4 + (2\mu K - 3B)p^2(4 - p^2)\delta + 2Bp(4 - p^2) + (4 - p^2)\left((\mu K - B)p^2 + 4\mu K - 2Bp\right)\delta^2 & \text{if } \frac{3B}{2K} \leq \mu \leq \frac{2B}{K}; \\ (\mu K - 2B)p^4 + (2\mu K - 3B)p^2(4 - p^2)\delta + 2Bp(4 - p^2) + (4 - p^2)\left((\mu K - B)p^2 + 4\mu K - 2Bp\right)\delta^2 & \text{if } \mu \geq \frac{2B}{K} \end{cases}$$

$$\equiv F(\delta).$$

Since $F'(\delta) \geq 0$ and therefore $F(\delta)$ is increasing function in $[0,1]$ and takes its maximum value at $\delta = 1$. (3.12) reduces to

$$C|a_2a_4 - \mu a_3^2| \leq \tag{3.13}$$

$$\begin{cases} (2B-\mu K)p^4+(3B-2\mu K)p^2(4-p^2)+(4-p^2)[(B-\mu K)p^2-4\mu K] & \text{if } \mu\le 0;\\ (2B-\mu K)p^4+(3B-2\mu K)p^2(4-p^2)+(4-p^2)[(B-\mu K)p^2+4\mu K] & \text{if } 0\le\mu\le\frac{B}{K};\\ (2B-\mu K)p^4+(3B-2\mu K)p^2(4-p^2)+(4-p^2)[(\mu K-B)p^2+4\mu K] & \text{if } \frac{B}{K}\le\mu\le\frac{3B}{2K};\\ (2B-\mu K)p^4+(2\mu K-3B)p^2(4-p^2)+(4-p^2)[(\mu K-B)p^2+4\mu K] & \text{if } \frac{3B}{2K}\le\mu\le\frac{2B}{K};\\ (\mu K-2B)p^4+(2\mu K-3B)p^2(4-p^2)+(4-p^2)[(\mu K-B)p^2+4\mu K] & \text{if } \mu\ge\frac{2B}{K} \end{cases}$$

$\equiv G(p)$. Then

$$C|a_2a_4-\mu a_3^2|\le \operatorname{Max} G(p). \qquad (3.14)$$

Case (i) $\mu\le 0$.

Then $G(p)=-2(B-\mu K)p^4+8(2B-\mu K)p^2-16\mu K$.

$G(p)$ attains its maximum value at $p=\sqrt{\frac{2(2B-\mu K)}{B-\mu K}}$ and $\max G(p)=\frac{8(2B-\mu K)^2}{B-\mu K}-16\mu K$. Putting the value of $\max G(p)$ in (3.14), we obtain (3.1).

Case (ii) $0\le\mu\le\frac{B}{K}$.

Then $G(p)=-2(B-\mu K)p^4+16(B-\mu K)p^2+16\mu K$.

Since $G'(p)\ge 0$, $\max G(p)=G(2)=32(B-\mu K)+16\mu K$. With this value of $\max G(p)$, (3.2) follows from (3.14).

Case (iii) $\frac{B}{K}\le\mu\le\frac{3B}{2K}$.

Then $G(p)=-8(\mu K-B)p^2+16\mu K$ which implies that $G(p)\le 16\mu K$.

Case (iv) $\frac{3B}{2K}\le\mu\le\frac{2B}{K}$.

Then $G(p)=-2(2\mu K-3B)p^4-8(2B-\mu K)p^2+16\mu K$.

In this case also $G(p)\le 16\mu K$.

Combination of cases (iii) and (iv) with (3.14) gives (3.3).

Case (v) $\mu\ge\frac{2B}{K}$.

Then $G(p)=-2(\mu K-B)p^4+8(\mu K-2B)p^2+16\mu K$.

It is easy to show that $G(p)$ is maximum at $p=\sqrt{\frac{2(\mu K-2B)}{\mu K-B}}$ and $\max G(p)=\frac{8(\mu K-2B)^2}{\mu K-B}+16\mu K$. Subtituting the value of $\max G(p)$ in (3.14), we arrive at (3.4). ■

Remark. Putting $\alpha=0$ in the theorem, we get the **Corollary 3.2.**

On the similar pattern as above, we can have the following result.

THEOREM 3.4 Let $f\in T_{1(s)}(\alpha)$, then

$$|a_2a_4-\mu a_3^2|\le\begin{cases}\frac{8(5B-\mu K)^2}{C(3B-\mu K)}-\frac{49\mu}{81(1+2\alpha)^2}, & \mu\le 0;\\ \frac{8(5B-2\mu K)^2}{C(3B-\mu K)}+\frac{49\mu}{81(1+2\alpha)^2}, & 0\le\mu\le\frac{5B}{2K};\\ \frac{49\mu}{81(1+2\alpha)^2}, & \frac{5B}{2K}\le\mu\le\frac{5B}{K};\\ \frac{8(\mu K-5B)^2}{C(\mu K-3B)}+\frac{49\mu}{81(1+2\alpha)^2}, & \mu\ge\frac{5B}{K},\end{cases}$$

where

$$B=27(1+2\alpha)^2$$
$$K=98(1+\alpha)(1+3\alpha)$$
$$C=2592(1+\alpha)(1+3\alpha)(1+2\alpha)^2$$

Remark. On taking $\alpha=0$ in the theorem, we obtain the **Corollary 3.4.**

References

1. C. Carathéodory,Uber den Variabilitatsbereich der Fourier'schen Konstanten von positive harmonischen Funktionen. *Rend. Circ. Mat. Palermo*, **32(**1911), 193-213.
2. R. N. Das &P. Singh, on subclasses of schlicht mapping, *Ind. J. Pure Appl. Math.* **8(**1977), 864-872.
3. P. L. Duren, *Univalent Functions*, Berlin, Springer-Verlag, 1983.
4. Harjinder Singh and Parvinder Singh, Estimates of second hankel determinant for subclasses of generalized pascu classes of functions, *Int. J of Recent Scientific Research*, Vol. 6, Isuue 12, pp. 7961-7967, Dec. 2015.
5. W. K. Hayman, *Multivalent function*, Camb. Tracts in Math. and Math. Phy., Camb. Univ. Press, Cambridge, 1958.
6. R. J. Libera and E. J. Zlotkiewiez, Early Coefficients of the Inverse of a Regular Convex Functions. *Proc. Amer. Math. Soc.* **85(**1982), 225-230.
7. J. W. NoonamandD. K. Thomas, Coefficient differences and Hankel determinant of a really p-valent functions, *Proc. London Math. Soc.* **25 (**1972), 503-524.
8. J. W. NoonamandD. K. Thomas, on the second Hankel determinant of a really p- valent functions, *Trans. Amer. Math. Soc.* (223) (1976): 337-346.
9. Ch. Pommerenke, on the coefficient and Hankel determinant of univalent functions, *J. London Math. Soc.* **41(**1966), 111-122.
10. Ch. Pommerenke, On the Hankel determinant of univalent functions, *Mathematika*, **14** (1967), 108-112.
11. K. Sakaguchi, on certain univalent mapping. *J. Math. Soc. Japan*, **11(**1959), 72-75.

****

How to cite this article:

Harjinder Singh. 2016, Generalised Second Hankel Determinant For A Class of Analytic Functions. *Int J Recent Sci Res.* 7(11), pp. 14356-14358.