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Research Article

DOUBLE DOMINATION NUMBER IN THE UNITARY ADDITION CAYLEY GRAPHS

Cristopher John S. Rosero

Mathematics Department, Cebu Normal University, Cebu City, Cebu, Philippines

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ABSTRACT

Let $(\Gamma,*)$ be a finite group and e be its identity. Let A be a generating set of Γ such that e A and $a^{-1} \in A$ for all $a \in A$. The Cayley graph is defined by $G = (V(G), E(G))$, where $V(G) = \Gamma$ and $E(G)=\{(x, x*a) \mid x \in V(G), a \in A\}$, denoted by $Cay(\Gamma,A)$. For a positive integer $n > 1$, the unitary addition Cayley graph G_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes the set of all units of the ring Z_n, then two vertices a, b are adjacent if and only if $a + b \in U_n$. In this paper, we attempt to find the *double domination number* of the unitary addition Cayley graphs G_n for some n.

INTRODUCTION

Let $G = (V(G), E(G))$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order of G*. The *open neighborhood* $N(v)=N_G(v)$ of a vertex *v* consists of the vertices adjacent to *v* and $d(v) = d_G(v) = |N(v)|$ is the degree of *v*. The *closed neighborhood* of a vertex $v \in V$ is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. A *regular graph* is a graph whose vertices have all the same degree. If $d(x) = r$ for all $x \in V(G)$, we call *G r-regular* and if $d(x) \in \{ r, r + 1\}$ we say that *G* is (r_1, r_2) - *semiregular*. The graph definitions, terminologies, and notations, unless otherwise indicated, are taken from the books by Harary in [7] and by Chartrand and Lesniak in [2].

A subset *S* of *V(G)* is a *dominating set* in *G* if $N_G[S] = S$ $N_G(S) = V(G)$ where $N_G(S) = \{v \in V(G) : xv \in E(G)$ for some $x \in S\}$. Equivalently, a subset *S* of *V(G)* is a dominating set in *G* if for every $v \in V(G)\backslash S$, there exists $x \in S$ such that $xv \in E(G)$. Furthermore, the minimum cardinality of a dominating set in *G*, denoted by $\gamma(G)$, is the *domination number* of *G*.

Let $G = (V(G), E(G))$ be a graph with no isolated vertices. A subset *S* of V(G) is a *double dominating set* of G if S dominates every vertex of G at least twice. The *double domination number* of G, denoted by $\gamma_{x2}(G)$, is the minimum cardinality of a double dominating set of G.

Let Γ be a finite group with *e* as the identity. A generating set of the group Γ is a subset *A* such that every element of Γ can be expressed as the product of finitely many elements of *A*. Assume that e A and $a \in A$ implies $a^{-1} \in A$. The *Cayley graph* $G = (V,E)$, where $V(G) = \Gamma$ and $E(G)=\{ (x,x*a) : x \in V(G), a \in A\}$, and it is denoted by $Cay(\Gamma,A)$. For any positive integer *n*, let Z_n denotes the additive cyclic group of integers modulo *n*. If we represent the elements of Z_n by $0, 1, \ldots, n-1$, then $U_n=\{ a \in Z_n : \gcd(a, n) = 1\}$ is a subset of Z_n of order $\varphi(n)$, where $\varphi(n)$ is the Euler's φ function. The Cayley graph $Cay(Z_n,U_n)$ is known as *unitary Cayley graph*. Hence, the *unitary Cayley graph* X_n is the graph whose vertex set is Z_n, the integers modulo *n* and if U_n denotes set of all units of the ring Z_n, then two vertices *a* and *b* are adjacent if and only if $a - b \in U_n$. The unitary Cayley graph X_n is also defined as, $X_n = Cay(Z_n, U_n)$ [1].

For a positive integer $n > 1$, the *unitary addition Cayley graph* $G_n = Cay^+(Z_n,U_n)$ is the graph whose vertex set is $Z_n = \{0, 1, 2, \quad ,n-1\}$ and the edge set $E(G_n)=\{ab : a, b \in Z_n, a + b \in U_n\}$ where $U_n = \{a \in Z_n : \gcd(a, n) = 1\}$ [3]. The graph G_n is regular if *n* is even and semi regular if *n* is odd [1].

In this paper, we attempt to find the double domination number of the U*nitary Addition Cayley graphs* G_n for some *n*.

*Corresponding author: **Cristopher John S. Rosero**
Mathematics Department, Cebu Normal University, Cebu City, Cebu, Philippines

PRELIMINARIES

Theorem 2.1 [6] For any graph G, $\gamma_{x2}(G) \leq n$.

Theorem 2.2 [5] For a cycle C_p with p vertices,

$$\gamma_{x2}(C_p) = \frac{2n}{3}.$$

Let $X_n = Cay(Z_n, U_n)$ be a unitary Cayley graph.

Theorem 2.3 [4] X_n is $\varphi(n)$ - *regular* for all n.

Theorem 2.4 [8] X_n, $n \geq 2$, is bipartite if and only if n is even.

Theorem 2.5 [1] The unitary addition Cayley graph G_n is connected for all n.

Theorem 2.6 [1] The unitary addition Cayley graph G_n is isomorphic to the unitary Cayley graph X_n if and only if n is even.

DOUBLE DOMINATION NUMBER IN UNITARY ADDITION CAYLEY GRAPHS

We now give some results of the double domination number in the Unitary Addition Cayley graphs G_n for some n.

Theorem 3.1 Let n be prime such that $n \geq 3$. Then $\gamma_{x2}(G_n) = 3$.

Proof: Suppose n is prime. Let v be a vertex of the unitary addition Cayley graph G_n. Since $|U_n| = \varphi(n)$, let $U_n = \{1, v_2, \ldots, v_{\varphi(n)-1}, v_{\varphi(n)}\}$. By the definition of unitary addition Cayley graph, $0 \in V(G_n)$ is adjacent to a vertex $v \in V(G_n)$ if and only if $(0 + v, n) = 1$. This implies that $(v, n) = 1$. That means, $v \in U_n$. Hence, $d(0) = \varphi(n)$. Since n is prime, then $d(0) = n$-1 which implies that 0 is adjacent to any vertex $v \in V(G_n)$.

Now, let $DD \subseteq V(G_n)$ be a double dominating set of G_n and $0 \in DD$. Consider $1 \in V(G_n)$. Since n is prime, then 1 is adjacent to any $u \in V(G_n)$ except n-1 since $1 + (n-1)$ U_n. If we let $1 \in$ DD, then DD dominates every vertex of G_n at least twice except $n\text{-}1 \in V(G_n)$. Hence $DD = \{0, 1, n\text{-}1\}$ and is a minimum double dominating set of G_n. Therefore, $\gamma_{x2}(G_n) = |DD| = 3$.

Theorem 3.2 Let n be an even integer such that $\varphi(n)=2$. Then $\gamma_{x2}(G_n) = \frac{2n}{3}$.

Proof: Suppose $\varphi(n)=2$. Then $|U_n| = 2$. Since n is even, then G_n is isomorphic to the Unitary Cayley graph X_n from Theorem 2.6. Hence, G_n is *2-regular* by Theorem 2.3. Since G_n is connected for all n by Theorem 2.5, we must have G_n C_n, a cycle with n vertices. Therefore, by Theorem 2.2,

$$\gamma_{x2}(G_n) = \frac{2n}{3}.$$

Theorem 3.3 If n is an integer such that $n = 2^r$, $r > 2$, then $\gamma_{x2}(G_n) = 4$.

Proof: Let $n = 2^r$, $r > 2$. Then U_n consists all the odd vertices of Z_n. Since n is even, then no two even labeled vertices are adjacent. This implies that even labeled vertices and odd labeled vertices form a bipartition of the vertex set. Since n is even, then G_n is $\varphi(n)$*-regular* by Theorem 2.6 and Theorem 2.3. Let $v \in V(G_n)$.Then $deg(v) = \varphi(n) = \varphi(2^r) = 2^r - 2^{r-1} = \frac{n}{2}$.

Now let DD be a double dominating set of G_n and $v_1, v_2 \in DD$. Without loss of generality, let v_1 and v_2 be even labeled vertices of G_n. Then v_1 and v_2 are adjacent to every odd vertex of G_n. Similarly, if we let v_3 and v_4 be elements of DD such that v_3 and v_4 are odd labeled vertices, then v_3 and v_4 are adjacent to every even labeled vertices of G_n. Hence, $DD = \{v_1, v_2, v_3, v_4\}$ is a minimum double dominating set of G_n. Therefore, $\gamma_{x2}(G_n) = |DD| = 4$.

Theorem 3.4 If $n = 2p$ where $p(\geq 5)$ is an odd prime, then $\gamma_{x2}(G_n) = 4$.

Proof: Let $n = 2p$. Then $|U_n| = \varphi(n) = p$-1. Since n is even, then due to Theorem 2.6 and Theorem 2.4, G_n is bipartite, i.e., even labeled vertices and the odd labeled vertices form a bipartition of the vertex set.

Let DD be a double dominating set of G_n. Now, let $v_1 \in DD$ such that v_1 is an odd labeled vertex in G_n. Since n is even, then $deg(v_1) = p$-1 by Theorem 2.6 and Theorem 2.3**.** Since the number of even labeled vertices is p, then there exists an even labeled vertex say v_2, such that $(v_1 + v_2, n) \neq 1$, that is, v_1 and v_2 are not adjacent. Thus, v_1 dominates itself and is adjacent to all even labeled vertices except v_2. Similarly, we can show that v_2 dominates itself and is adjacent to all odd labeled vertices except v_1 so that $v_2 \in DD$. Now choose $v_3 \in V(G_n)$, such that v_3 is an odd labeled vertex in G_n, to be an element of DD so that v_3 dominates itself, v_2 and other even labeled vertices of G_n except for an even labeled vertex say v_4. Hence, DD dominates all the even labeled vertices of G_n at least twice except v_4. Choose $v_4 \in V(G_n)$ to be an element of DD, so that all the even labeled vertices of $V(G_n)$ is dominated by at least two vertices in DD.

Note that, since v_4 is an even labeled vertex, then v_4 dominates itself and is adjacent to all odd labeled vertices except v_3. In the same manner, DD dominates all the odd labeled vertices of $V(G_n)$ at least twice.

Thus, $DD = \{v_1, v_2, v_3, v_4\}$ and DD is a minimum double dominating set of G_n. Therefore, $\gamma_{x2}(G_n) = |DD| = 4$.

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