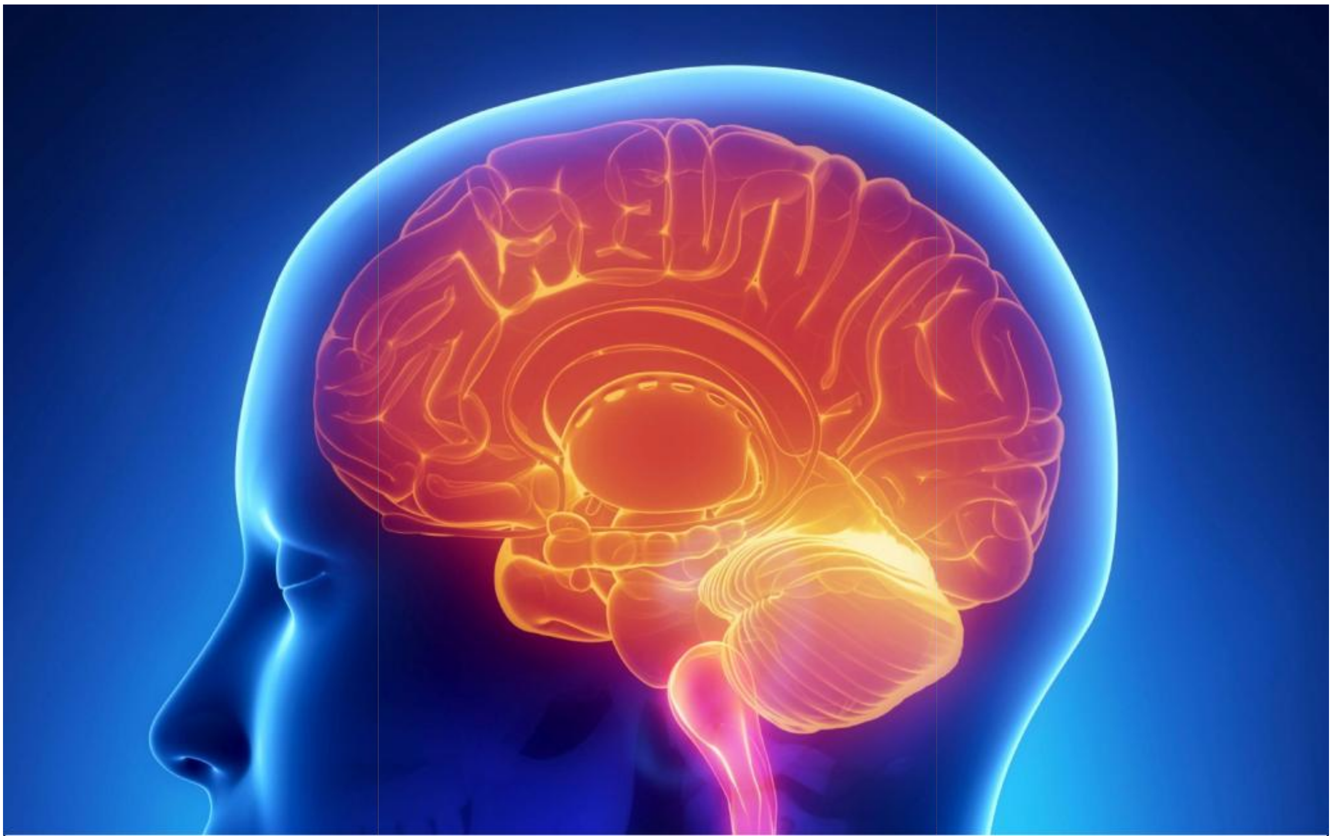

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RESEARCH ARTICLE

ON INVARIANT RANDOM BEST APPROXIMATIONS

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ABSTRACT

In this paper random fixed point results are proved for random Banach operator which is defined on separable closed subset of a complete p-normed space .As application , these results are used to study the random best approximations. This work extends or provides stochastic versions of several well-known results.

INTRODUCTION

The random approximations and random fixed point theorems are stochastic generalizations of usual approximations and fixed points theorems. Recently, many researchers are interesting in this subject such as (Beg and Shahzad, 1994, 2000) who proved some theorems about the best approximations as applications for some stochastic fixed point theorems. Also, (Khan *et al*, 2002) proved random fixed point theorems for non-expansive random operators defined on a class of non-convex sets containing the subclass of star-shaped sets in locally bounded topological vector spaces, and then, they obtained Brosowski – Memardus type theorems about random invariant of approximation (for Brosowski – Memardus type see (Kuratowski,1965) or (Meinardus,1963) (Nashine, 2008) established the existence of random fixed point as random best approximations with respect to compact and weakly compact domain. (Alsaidy *et al*, 2015) proved coincidence point results for pair of commuting mapping defined on weakly compact separable subset of complete p-normed space. And then, use them to study the random best approximation in p-normed space with separablity condition. In this paper, some results regarding random best approximations are proved as a consequence of random fixed point theorems in p-normed space whose dual separates the points of X.

Preliminaries

We need the following definitions and facts

Let X be a linear space and $\|.\|_p$be a real valued function on X with $0 < p \leq 1$. The pair $(X, \|.\|_p)$ is called a p-normed space if for all for all x , y in X and scalars λ :

1. $\|x\|_p \geq 0$ and $\|x\|_p = 0$ iff x = 0.
2. $\|\lambda x\|_p = |\lambda|^p \|x\|_p$
3. $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

Therefore, every p-normed space X is a metric space with$d(x, y) = \|x - y\|_p$, for all x , y in X . If $p = 1$, we have the concept of a normedspace (Nashine, 2006).

It is known that the topology of a Hausdorff locally bounded topological vector space is obtained by some p-norms, 0< $p \leq 1$. (Nashine, 2006) the spaces $L_p[0,1]$ andl_p, 0< $p \leq 1$ are p-normed space for other examples see (Nashine, 2006). We say that the continuous dual $X^`$of $X$ is separating if for each$0 \neq x \in X$, there exists $f \in X^`$such that $f(x) \neq 0$. If this

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satisfied then the weak topology of X will be Hausdorff (Nashine, 2008).A p-normed space is not necessarily locally convex space (Nashine, 2008). And the continuous dual $X^{`}$ of p-normed space X need not separate the point of X (Nashine, 2008).

For a subset A of X and $x_\circ \in X$, define the set $p_A(x_\circ) = \{z \in A: \|x_\circ - z\|_p\}$.

Then an element $z \in p_A(x_\circ)$ is called a best approximation of $x_\circ$ in A (Nashine, 2009), where $d(x_\circ, A) = \inf_{a\in A}\{\|x_\circ - a\|_p\}$(Nashine, 2009). The set A is called starshaped (Nashine, 2009) if there exists at least one point $q \in A$ such that $kx + (1-k)q \in A$ for all $x \in A$ and $0 \le k \le 1$. In this case q is called the starcenter of A.

The mapping $h: A \to X$ is called

i. nonexpansive (Zeidler,1986) if $d(h(x), h(y)) \le d(x, y)$. for each $x, y \in A$.
ii. ii. Banach operator (Shahzad,1995)if there exists a constant k ($0 \le k < 1$) such that $d(h(x), h^2(x)) \le kd(x, h(x))$. for each $x \in A$.
iii. compact (Zeidler,1986)if , for any nonempty bounded subset S of A , $\overline{h(S)}$ is compact set in X

The following classes are needed

CB(X) is the classes of all bounded closed subsets of X,

K(X)is the classes of all nonempty compact subsets of X,

Also, $\bar{A}$ is denoted to the closure of a set A.

Now, let the pair (Ω,Σ) with be denote to the measurable space with Σ a sigma algebra of subsets of Ω.

Definition (2.1)(O'regan et al ,2003)

A mapping $F: \Omega \to 2^X$is called (Σ-) measurable (respectively, weakly measurable) if, for any closed (respectively, open) subset B of X , $F^{-1}(B) = \{\omega \in \Omega: F(\omega) \cap B \neq \emptyset\} \in \Sigma$.

Definition (2.2) (O'regan et al, 2003)

A measurable mapping $\delta: \Omega \to X$ is called a measurable selector of a measurable mapping $F:\Omega \to 2^X$if $\delta(\omega) \in F(\omega)$ for each $\omega \in \Omega$.

Definition (2.3) (Brosowski, 1969)

A mapping $h: \Omega \times A \to X$ is called a random operator if for any $x \in A, h(., x)$.

Definition (2.4)(Nashine,2008)

A measurable mapping $\delta: \Omega \to A$ is called random fixed point of a random operator $h: \Omega \times A \to X$ if for every$\omega \in \Omega$, $\delta(\omega) = h(\omega, \delta(\omega))$.

Definition (2.5) (Shahzad, 1995)

A random operator $h: \Omega \times A \to X$ is called continuous (non-expansive (compact, Banach operator, etc.) if for each $\omega \in \Omega$, $h(\omega, .)$ is continuous (non-expansive, compact, Banach operator, etc.).

Throughout this paper X will be p-normed space whose dual separates the points of X.

MAIN RESULTS

Firstly, we prove that

Theorem (3.1)

Let X be a complete p-normed space. A continuous Banach random mapping $h: \Omega \times A \to A$, where A is a separable closed subset of X , has a random fixed point.

Proof:

Suppose that $Q_\circ: \Omega \to A$ is measurable mapping. since h is Banach operator , then by induction that

$$\left\|h^{n+1}(\omega, Q_\circ(\omega)) - h^n(\omega, Q_\circ(\omega))\right\|_p \le k^n(\omega)\left\|Q_\circ(\omega) - h(\omega, Q_\circ(\omega))\right\|_p, \text{ for all } \omega \in \Omega \text{ and n= } 1,2,.$$

Let $Q_1(\omega) = h(\omega, Q_\circ(\omega))$ then by Theorem (6.5) (Himmelberg, 1975) the mapping $Q_1$is measurable and the sequence of measurable mappings $Q_n(\omega)$can be defined as follows:

$$Q_n(\omega) = h(\omega, Q_{n-1}(\omega)) = h^n(\omega, Q_\circ(\omega)) \text{ for all } \omega \in \Omega \text{ and n= } 1,2, \dots$$

Assume that $n \le m$, then for all $\omega \in \Omega$, we have

$$\|Q_n(\omega) - Q_m(\omega)\|_p = h^n(\omega, Q_\circ(\omega)) - h^m(\omega, Q_\circ(\omega))_p$$

$$\le k^n(\omega)\left\|Q_\circ(\omega) - h(\omega, Q_{m-n-1}(\omega))\right\|_p$$

$$=k^n(\omega)\|Q_\circ(\omega) - Q_{m-n}(\omega)\|_p$$

$$\le k^n(\omega)[\|Q_\circ(\omega) - Q_1(\omega)\|_p + \|Q_1(\omega) - Q_2(\omega)\|_p + \cdots \|Q_{m-n-1}(\omega) - Q_{m-n}(\omega)\|_p]$$

$$=k^n(\omega)\|Q_\circ(\omega) - Q_1(\omega)\|_p[1 + k(\omega) + \cdots + k^{m-n-1}(\omega)]$$

$$< \frac{k^n(\omega)}{1-k(\omega)}\|Q_\circ(\omega) - Q_1(\omega)\|_p$$

Since $0 < k(\omega) < 1$ for all $\omega \in \Omega$, then $\{Q_n(\omega)\}$ is a Cauchy sequence in$A$. Since X is a complete metric space which induced by $\| \ \ \|_p$and A is a closed subset of$X$, hence by (Kreyzig, 1978) A is complete metric space.

This implies $\{Q_n(\omega)\}$ converges to $Q(\omega) \in A$. By continuity of h we have $\lim_{n\to\infty} h(\omega, Q_n(\omega)) = h(\omega, Q(\omega))$ for all $\omega \in \Omega$. Therefore the mapping $Q: \Omega \to A$ is a random fixed point of h.■

Remark (3.2)

Theorem (3.1) remains true if A is a separable, closed subset of a p- normed space X and $\overline{h(\omega, A)}$ is compact, for any $\omega \in \Omega$.

Definition (3.3)

Let X be a p-normed space, $A \subseteq X$ and *and* $h: \Omega \times X \to X$ be a random operator we say that A has property (P) if

i. $h: \Omega \times A \to A$
ii. $(1 - k_n)q + k_n h(\omega, x) \in A$, for some $q \in A$ and a fixed real sequence $< k_n >$ converging to 1 $(0 < k_n < 1)$ and for each $x \in A$ and for each $\omega \in \Omega$.

Theorem (3.4)

If $h: \Omega \times A \to A$ is nonexpansive random mapping , where A is a separable closed and has property (P) subset of p-normed space X, and $\overline{h(\omega, A)}$ is compact , for any $\omega \in \Omega$, then h has a random fixed point.

Proof

Since h is defined on $\Omega \times A$ in to A and A has property (P), then $(1 - k_n)q + k_n h(\omega, x) \in A$, for some $q \in A$ and a fixed real sequence $< k_n >$ converging to 1 $(0 < k_n < 1)$, for all $x \in A$ and for all $\omega \in \Omega$.

For $n \geq 1$, defined the random mappings G_n by

$G_n(\omega, x) = (1 - k_n)q + k_n h(\omega, x)$, for all $x \in A$ and all $\omega \in \Omega$.

This implies to, $G_n : \Omega \times A \to A$.

Since h is nonexpansive random mapping, then each G_n is Banach operator as follows:

$$\|G_n(\omega, x) - G_n{}^2(\omega, x)\|_p = \|G_n(\omega, x) - G_n(\omega, G_n(\omega, x))\|_p$$

$$= \|(1 - k_n)q + k_n h(\omega, x) - [(1 - k_n)q + k_n h(\omega, G_n(\omega, x))]\|_p$$
$$= (k_n)^p \|h(\omega, x) - h(\omega, G_n(\omega, x))\|_p$$
$$\leq (k_n)^p \|x - G_n(\omega, x))\|_p$$

By continuity of h and $\overline{h(\omega, A)}$ is compact for all $\omega \in \Omega$, we have each G_n is continuous random mappings and $\overline{G_n(\omega, A)}$ is compact for each $\omega \in \Omega$
So, by remark (3.2) there is a measurable map $\delta_n: \Omega \to A$ such that $\delta_n(\omega) = G_n(\omega, \delta_n(\omega))$.

Now, for each n Define $Q_n: \Omega \to k(A)$ by

$Q_n(\omega) = \overline{\{\delta_\iota(\omega): \iota \geq n\}}$.

Also, define $Q: \Omega \to k(A)$ by $Q(\omega) = \cap_{n=1}^{\infty} Q_n(\omega)$.

Then by theorem (4.1) (Kreyzig, 1978) Q is measurable, this implies Q has measurable selector δ (Khan and Khan, 1995). Fix any $\omega \in \Omega$. Since $\overline{h(\omega, A)}$ is compact and $h(\omega, \delta_n(\omega)) \subseteq \overline{h(\omega, \delta_n(\omega))}$, then there is a subsequence $\{h(\omega, \delta_n(\omega)\}$ convergent to $\delta(\omega)$;

Since $k_m \to 1$ and

$\delta_m(\omega) = G_m(\omega, \delta_m(\omega)) = (1 - k_m)q + k_m h(\omega, \delta_m(\omega))$ converges to $\delta(\omega)$.

Since h is continuous random mapping, then $h(\omega, \delta_m(\omega))$ converges to $h(\omega, \delta(\omega))$.

This implies $h(\omega, \delta(\omega)) = \delta(\omega)$, for all $\omega \in \Omega$.

Therefore $\delta(\omega)$ is random fixed point of h. ■

As a consequence we obtain

Corollary (3.5)[(Beg and Shahzad ,1994) ,Theorem 3]

If $h: \Omega \times A \to A$ is nonexpansive random operator,

where A is a nonempty separable closed and starshaped subset of normed space X, and $\overline{h(\omega, A)}$ is compact , for any $\omega \in \Omega$, then h has a random fixed point.

Now, we apply the above results to get an invariant best approximation

Remark (3.6) (Beg and Shahzad, 1994)

If a mapping $h: X \to X$ leaves a subset A of X invariant, then in the following $h_{|A}$ is denoted to restriction of h to A .

Theorem (3.7) (Khan and Khan, 1995)

Let X be a metric space and $h: X \to X$ a nonexpansive mapping with a fixed point $x_\circ \in X$. If A is closed h-invariant of X and $h_{|A}$ compact mapping, then the set $p_A(x_\circ)$ of best approximation is nonempty.

Theorem (3.8)

Let A be a nonempty closed subset of a p-normed space X and $h: \Omega \times X \to X$ is a nonexpansive random mapping with $h(\omega, x_\circ)$

$= x_\circ$ for and $h(\omega, A) \subseteq A$ for all $\omega \in \Omega$. with $h(\omega, .)_{|A}$ compact If $p_A(x_\circ)$ is separable and has the property (P) , then the point $x_\circ$ has best random approximation which is also a random fixed point of h.

Proof

Let M $= p_A(x_\circ)$, by theorem (3.7), M is nonempty that is there exists $x \in A$ such that $d(x_\circ, A) = \|x - x_\circ\|_p$. Since h is nonexpansive random operator and $h(\omega, x_\circ) = x_\circ$, hence

$$d(x_\circ, A) = \|x_\circ - x\|_p \geq \|h(\omega, x_\circ) - h(\omega, x)\|_p$$
$$= \|x_\circ - h(\omega, x)\|_p$$

This implies $\|x_\circ - h(\omega, x)\|_p \leq d(x_\circ, A)$ for all $x \in M$. thus, M

is $h(\omega, .)$-invariant.

Since A is a closed subset of X and $h(\omega, .)_{|A}$ is compact, then M is a closed subset of A and $\overline{h(\omega, M)}$ is compact

Hence all conditions of Theorem (3.4) are satisfying on M
Thus there exists a measurable mapping $\delta: \Omega \to M$ such that $\delta(\omega) = h(\omega, \delta(\omega))$ that is there exists best random approximation which is also a random fixed point of h.

Corollary (3.9)

If $h: \Omega \times A \to A$ is nonexpansive random mapping, where A is a separable closed and starshaped subset of complete p-normed space X , then h has a random fixed point.

Corollary (3.10)

If $h: \Omega \times A \to A$ is nonexpansive random operator, where A is a separable closed and starshaped subset of a p-normed space X and $\overline{h(\omega, A)}$ is compact, for any $\omega \in \Omega$, then h has a random fixed point.

Corollary (3.11) [(Beh and Shahzad, 1994), Theorem 4]

Let X be a normed space. If $h: \Omega \times X \to X$ is a nonexpansive random mapping with $h(\omega, x_\circ) = x_\circ$ for any $\omega \in \Omega$, leaving a subspace A of X invariant,$h(\omega, .)_{|A}$ is compact and and$p_A(x_\circ)$ is separable, then the point $x_\circ$ has best random approximation which is also a random fixed point of h.

Corollary (3.12)[(Beh and Shahzad,1994) ,Theorem 5]

Let $h: \Omega \times X \to X$ be a nonexpansive random mapping with$h(\omega, x_\circ) = x_\circ$ for any$\omega \in \Omega$, leaving convex subset A of X invariant and$h(\omega, .)_{|A}$ compact. If $p_A(x_\circ)$ is nonempty and separable, then the point $x_\circ$ has best random approximation which is also a random fixed point of h.

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