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RESEARCH ARTICLE

ON THE TERNARY CUBIC DIOPHANTINE EQUATION $7x^2 - 4y^2 = 3z^3$

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ABSTRACT

The ternary cubic Diophantine equation given by $7x^2 - 4y^2 = 3z^3$ is analyzed for its non-zero distinct integer points on it. Different patterns of integer points for the equation under consideration are obtained. A few interesting relations between solutions and special numbers are obtained.

INTRODUCTION

The ternary cubic diophantine equation offer an unlimited field of research due to their variety [Carmichael. R.D (1959), Dickson. L.E (2005), Mordell. L.J (1969)]. For an extensive of various problems, one may refer [Gopalal. M.A (2010,2011,2012,2013a,b,c,2014)]. This communication concerns with yet another interesting ternary cubic diophantine equation $7x^2 - 4y^2 = 3z^3$ for determining its infinitely many non-zero integral points. Also, a few interesting relations between the solutions and special numbers are presented.

Notations

$CP_{m,n}$ - Centered Pyramidal number of rank n with size m

$T_{m,n}$ - Polygonal number of rank n with size m.

$\Pr_n$ - Pronic number of rank n.

P_n^m -Pyramidal number of rank n with size m.

Method of Analysis

The Diophantine equation to be solved for its non-zero distinct integral solution is,

$$7x^2 - 4y^2 = 3z^3 \tag{1}$$

To start with, it is seen that (1) is satisfied by the following triples of integers:

$$(1,1,1),\quad (-3,6,-3),\quad (-19,76,-19),\quad (\sim^3,\sim^3,\sim^2),$$
$$\left(\frac{7.2^{3k}-2^{k+2}}{3},\frac{7.4^k-4}{3},\frac{7.4^k-4}{3}\right)$$

In what follows, we illustrate methods of obtaining non-zero distinct integer solutions to (1).

The substitution of linear transformations

$$x = 2X_1 + 4T,\ y = 2X_1 + 7T,\ z = 2Z_1 \tag{2}$$

in (1) leads to

$$X_1^2 - 7T^2 = 2Z_1^3 \tag{3}$$

Assume $Z_1 = Z_1(a,b) = a^2 - 7b^2; a,b > 0$ (4)

(3) is solved through different approaches and different patterns of solutions thus obtained for (1) are illustrated below:

Method: 1

Write 2 as $2 = \left(3+\sqrt{7}\right)\left(3-\sqrt{7}\right)$ (5)

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Using (4) and (5) in (3) and employing the method of factorization it is written as

$$\left(X_1+\sqrt{7}T\right)\left(X_1-\sqrt{7}T\right)=\left(3+\sqrt{7}\right)\left(3-\sqrt{7}\right)\left(a+\sqrt{7}b\right)^3\left(a-\sqrt{7}b\right)^3$$

which is equivalent to the system of equations

$$\left.\begin{aligned}\left(X_1+\sqrt{7}T\right)&=\left(3+\sqrt{7}\right)\left(a+\sqrt{7}b\right)^3\\ \left(X_1-\sqrt{7}T\right)&=\left(3-\sqrt{7}\right)\left(a-\sqrt{7}b\right)^3\end{aligned}\right\} \tag{6}$$

Equating the rational and irrational parts in (6), we have

$$X_1 = 3a^3 + 63ab^2 + 21a^2b + 49b^3$$
$$T = a^3 + 21ab^2 + 9a^2b + 21b^3$$

Employing (2), the values of x,y and z satisfying (1) are given by

$$x = x(a,b) = 10a^3 + 210ab^2 + 78a^2b + 182b^3$$
$$y = y(a,b) = 13a^3 + 273ab^2 + 105a^2b + 245b^3$$
$$z = z(a,b) = 2a^2 - 14b^2$$

Properties

1. $x(2.b) + z(2,b) - 182CP_{6,b} - 406t_{4,b} \equiv 88 \pmod{312}$
2. $y(2,a) - x(2,a) - 126P_a^5 - 63t_{4,a} \equiv 24 \pmod{108}$
3. $y(2,a) - 490P_a^5 - 301\mathrm{Pr}_a \equiv 104 \pmod{119}$
4. $3[-z(a,a)]$ represents a perfect square.
5. Each of the following expressions represents a cubical integer:

A. $3042[y(a,a) - x(a,a)]$
B. $5766[y(a,a) + x(a,a)]$

***Method*: 2**

One may write (3) as

$$X_1^2 - 7T^2 = 2Z_1^3 * 1 \tag{7}$$

Write 1 as $1 = \dfrac{\left(4+\sqrt{7}\right)\left(4-\sqrt{7}\right)}{9}$ (8)

Using (4), (5) and (8) in (7); employing the method of factorization and equating positive factors, we get

$$\left(X_1+\sqrt{7}T\right)=\left(3+\sqrt{7}\right)\frac{\left(4+\sqrt{7}\right)}{3}\left(a+\sqrt{7}b\right)^3 \tag{9}$$

Equating rational and irrational parts of (9), we have

$$X_1 = \frac{1}{3}\left[19a^3 + 399ab^2 + 147a^2b + 343b^3\right]$$
$$T = \frac{1}{3}\left[7a^3 + 147ab^2 + 57a^2b + 133b^3\right]$$

As our aim is to find integer solutions choosing a=3A , b=3B we obtain as follows:

$$X_1 = 171A^3 + 3591AB^2 + 1323A^2B + 3087B^3$$
$$T = 63A^3 + 1323AB^2 + 513A^2B + 1197B^3$$
$$Z_1 = 9A^2 - 63B^2$$

In view of (2), the integer solutions of (1) are given by

$$x = x(a,b) = 594A^3 + 12474AB^2 + 4698A^2B + 10962B^3$$
$$y = y(a,b) = 783A^3 + 16443AB^2 + 6237A^2B + 14553B^3$$
$$z = z(a,b) = 18A^2 - 126B^2$$

Properties

1. $y(A,1) - x(A,1) - 1134P_A^3 - 972t_{4,A} \equiv 0 \pmod{3591}$
2. $x(A,1) - z(A,1) - 1188\ P_A^5 - 4086\ \mathrm{Pr}_A \equiv 2700 \pmod{8388}$
3. $y(A,A) - x(A,A) - 66744CP_{6,A} = 0$
4. $849[y(A,A) - x(A,A)]$ is a cubical integer.
5. $2[-z(A,A)]$ represents a nasty number.

Note

It is worth to note that 2 in (5) and 1 in (8) are also represented in the following ways:

$$2 = \frac{\left(5+\sqrt{7}\right)\left(5-\sqrt{7}\right)}{9} \qquad 2 = \frac{\left(27+\sqrt{7}\right)\left(27-\sqrt{7}\right)}{361}$$
$$2 = \frac{\left(13+\sqrt{7}\right)\left(13-\sqrt{7}\right)}{81} \qquad 1 = \left(8+3\sqrt{7}\right)\left(8-3\sqrt{7}\right)$$

By introducing the above representations instead of (5) and (8), one may obtain 6 different distinct integer solutions to (1).

Remarkable Observation

Let $\left(x_0, y_0, z_0\right)$ be any given integer solution to (1). In what follows, we illustrate a process of obtaining a general form of integer solutions to (1) based on the given solution.

Let
$$\begin{aligned} x_1 &= -3^3 x_0 + h \\ y_1 &= 3^3 y_0 + h \\ z_1 &= 3^2 z_0 \end{aligned} \tag{10}$$

be the second solution of (1)

Substituting (10) in (1), and simplifying, we get

$$h = 126x_0 + 72y_0$$

And thus, $$x_1 = 99x_0 + 72y_0 \qquad y_1 = 126x_0 + 99y_0 \tag{11}$$

The matrix representation of (11) is

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 99 & 72 \\ 126 & 99 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

The repetition of the above process leads to the general solution of (1) to be

$$x_n = Y_n x_0 + 2X_n y_0$$

$$y_n = Y_n y_0 + \frac{7}{2} X_n x_0$$

$$z_n = 3^{2n} z_0$$

where (X_n, Y_n) is the general solution of the Pellian equation $y^2 = 7x^2 + 27^2$

CONCLUSION

In this paper, we have presented different sets of non-zero distinct integer solutions to the ternary cubic equation $7x^2 - 4y^2 = 3z^3$.

As the cubic diophantine equations are rich in variety, one may search for other choices of equations along with their solutions and relations among the solutions.

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