In this paper, we give some results for differential subordination and superordination for analytic and multivalent functions in the open unit disk involving linear operator \( \frac{1}{g(z)} \). These results are obtained by investigating appropriate classes of admissible functions.

**Key words:** Analytic functions, differential subordination, superordination, multivalent function, Hadamard product (convolution).

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**INTRODUCTION**

**Preliminaries**

Let \( H(U) \) denote the class of analytic functions in the open unit disk \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \) and let \( H(\alpha, n) \) denote the subclass of \( H(U) \) of the form \( f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \cdots \), where \( a \in \mathbb{C} \) and \( n \in \mathbb{N} \) with \( n = H(0, 1) \) and \( H_1 = H(1, 1) \). If \( f, g \) are members of \( H(U) \) we say that a function \( f \) is subordinate to a function \( g \) or \( g \) is said to be superordinate to \( f \) if there exists a Schwarz function \( w(z) \) which is analytic in \( U \), with \( w(0) = 0 \) and \( |w(z)| < 1 \) for all \( z \in U \), such that \( f(z) = g(w(z)) \). In such a case we write \( f \prec g \). Further, if the function \( g \) is univalent in \( U \) then we have the following equivalent (see [3, 9]):

\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

Let \( \mathcal{A}(p) \) denote the class of all analytic functions of the form

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (z \in U, \ p \in \mathbb{N} = \{1, 2, 3, \ldots\}).
\] (1.1)

For function \( g \in \mathcal{A}(p) \) given by \( g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \), \( (p \in \mathbb{N}) \), the Hadamard product (or convolution) of \( f \) and \( g \) is defined by

\[
(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z).
\]

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For function $f, g \in A(p)$, we defined the linear operator $F_{n, k,p}^\ast : A(p) \to A(p)$ ($\lambda \geq 0, m \in N_0 = N \cup \{0\}$) by $F_{n, k,p}^0(f \ast g)(z) = (f \ast g)(z),
\begin{equation}
F_{n, k,p}^1(f \ast g)(z) = F_{n, k,p}(f \ast g)(z) = (1 - \lambda)(f \ast g)(z) + \frac{\lambda z}{p}(f \ast g)(z),
\end{equation}
and
\begin{equation}
F_{n, k,p}^m(f \ast g)(z) = F_{n, k,p}(F_{n, k,p}^{m-1}(f \ast g)(z))
\end{equation}
therefore, it can be easily seen that
\begin{equation}
F_{n, k,p}^m(f \ast g)(z) = F_{n, k,p}^{m+1}(f \ast g)(z) - (1 - \lambda)F_{n, k,p}^m(f \ast g)(z), \quad (\lambda \geq 0).
\end{equation}
From (1.2) we can easily deduce that
\begin{equation}
\frac{\lambda z}{p}(F_{n, k,p}^m(f \ast g)(z)) = F_{n, k,p}^{m+1}(f \ast g)(z) - (1 - \lambda)F_{n, k,p}^m(f \ast g)(z), \quad (\lambda > 0).
\end{equation}
The operator $F_{n, k,p}^m(f \ast g)$ was introduced and studied by Selvaraj and Selvakumaran [14], Aouf and Mostafa [2] and for $m = 1$, was introduced by Aouf and Mostafa [1].

Remark

1. Taking $m = 0$ and $b_k = \frac{(\alpha_i)_{k-1} \cdots (\alpha_g)_{k-1}}{(\beta_i)_{k-1} \cdots (\beta_g)_{k-1}} (\alpha_i, \beta_j \in \mathbb{C}, (i = 1, 2 \ldots q), (j = 1, 2 \ldots s)$.
2. If $q \leq s + 1, q, s \in N_0$ in (1.2), the operator $F_{n, k,p}^m(f \ast g)$ reduces to the Dziok-Srivastava operator $H_{p,q,s}(\lambda)$ which generalized many other operator (see [6]).
3. Taking $m = 0$ and $b_k = \frac{(\alpha_i)_{k-1} \cdots (\alpha_g)_{k-1}}{(\beta_i)_{k-1} \cdots (\beta_g)_{k-1}} (\lambda > 0; \beta \in N ; l, n \in N_0)$ in (1.2), the operator $F_{n, k,p}^m(f \ast g)$ reduces to Catas operator $F_{n, k,p}^m(l, \lambda)$ which generalized many other operators (see [4]).
4. The method of differential subordinations (also known as the admissible functions method) was introduced by Miller and Mocanu [7,8] and developed in [9,10].
5. Let $\Omega$ and $\Delta$ be any sets in $\mathbb{C}$ and let $p$ be an analytic function in the unit disk $U$ with $p(0) = a$ and let $\psi(r,s,t;\lambda) : \mathbb{C}^3 \times U \to \mathbb{C}$.
6. If $\psi(p(z), zp'(z), z^2p'(z); z), (z \in U) \subset \Omega \Rightarrow p(U) \subset \Delta$. In [10] the authors introduce the dual problem of the differential subordination which they call differential superordination.
7. If $\{\psi(p(z), zp'(z), z^2p'(z); z), (z \in U)\} = \Delta \Rightarrow p(U) = 0(U)$.

Definition 1.1 [9] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the (second – order) differential subordination,
\begin{equation}
\{\psi(p(z), zp'(z), z^2p'(z); z), (z \in U)\} < h(z),
\end{equation}
then $p$ is called a solution of differential subordination. The univalent function $q$ is called a dominant, if $p < q$ for all $p$ satisfying (iii).

Definition 1.2 [10] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h$ be analytic in $U$. If $p$ and $\psi(p(z), zp'(z), z^2p'(z); z)$ are univalent in $U$ and satisfy the (second – order) differential superordination.
\begin{equation}
h(z) < \psi(p(z), zp'(z), z^2p'(z); z), \quad (z \in U) \Rightarrow p(U) \subset \Delta.
\end{equation}

To prove our results, we need the following definitions and Lemmas.

Definition 1.3 [9] Denote by $Q$ the set of all functions $q$ that are analytic and injective on $U \setminus E(q)$, where
\[ E(q) = \left\{ \xi \in \partial U : \lim_{z \to \xi} q(z) = \infty \right\}, \]

and \( a \) is such that \( q^{(a)}(\xi) = 0 \) for \( \xi \in \partial U \setminus \xi \). Further let the subclass of \( Q \) for which \( q(0) = a \) be denoted by \( Q(a), Q(0) = Q_0 \) and \( Q(1) = Q_1 \).

**Definition 1.4** [9] Let \( \Delta \) be a set in \( \mathbb{C} \); \( q \in Q \) and \( n \) be appositive integer. The class of admissible functions \( \Psi_n[\Delta, q] \) consists of those functions \( \psi : \mathbb{C}^3 \times U \to \mathbb{C} \) that satisfy the admissibility condition:

\[ \psi(r, s, t; z) \not\in \Omega, \]

wherever \( r = q(\xi), s = k_0 q(\xi), \)

\[ Re \left\{ \frac{\xi + 1}{k_0} \right\} \geq k_0 \left\{ \left\lfloor 1 + \frac{q(\xi)}{q(\xi)} \right\rfloor \right\}, \]

where \( z \in U, \xi \in \partial U \setminus E(q) \) and \( k \geq n \). We write \( \Psi_1[\Delta, q] = \Psi[\Delta, q] \).

In particular, when \( q(z) = M \frac{Mz + \alpha}{Mz + \beta} \) with \( M > 0 \) and \( |\alpha| < M \), then \( q(U) = U_M = \{ w : |w| < M \}, q(0) = 0, E(q) = \emptyset \) and \( q \in Q \). In this case, we set \( \Psi_n[\Delta, M, \alpha] = \Psi[\Delta, q] = \Psi_1[\Delta, q] \), and in the special case when the set \( Q = U_M \), the class is simply denoted by \( \Psi_n[M, \alpha] \).

**Definition 1.5**[10] Let \( \xi \) be a set in \( \mathbb{C} \); \( q(z) \in H[\alpha, n] \) with \( q(z) \neq 0 \). The class of admissible functions \( \Psi[\xi, q] \) consist of this functions \( \psi : \mathbb{C}^3 \times U \to \mathbb{C} \) that satisfy the admissibility condition:

\[ \psi(r, s, t; z) \not\in \Omega, \]

wherever \( r = q(\xi), s = k_0 q(\xi), \)

\[ Re \left\{ \frac{\xi + 1}{k_0} \right\} \leq k_0 \left\{ \left\lfloor 1 + \frac{q(\xi)}{q(\xi)} \right\rfloor \right\}, \]

when \( z \in U, \xi \in \partial U \) and \( b \geq n \geq 1 \). In particular, we write \( \Psi_1[\xi, q] = \Psi[\xi, q] \).

**Lemma 1.1** [9] Let \( \psi \in \Psi_n[\Omega, q] \) with \( q(0) = a \). If the analytic function \( g(z) = a + a_n + a \sum_{n+1}^\infty \) satisfies

\[ \psi(g(z), zg'(z), z^2 g''(z); z) \in \Omega, \]

Then

\[ g(z) \subset q(z), \quad (z \in U). \]

**Lemma 1.2** [10] Let \( \psi \in \Psi_n[\Omega, q] \) with \( q(0) = a, g \in Q(a) \) and \( \psi(g(z), zg'(z), z^2 g''(z); z) \) is univalent in \( U \), then

\[ \Omega \subset \{ \psi(g(z), zg'(z), z^2 g''(z); z); (z \in U) \}, \]

implies

\[ q(z) \subset g(z). \]

In fact, the study of the class of admissible functions was revived recently by Mustafa and Darus [11] and Cho [5]. A similar problem for analytic functions was studied by many others for example see [10, 12, 13].

The object of the present paper, we give some results for differential subordination and superordination for multivalent function involving the linear operator \( \mathcal{L} \).

**Differential subordination results associated with linear operator**

**Definition 2.1** Let \( \Omega \) be a set in \( \mathbb{C}, q \in Q_0 \cap H[0, p] \). The class of admissible functions \( \Psi_n[\Delta, q] \) consists of those functions \( \phi : \mathbb{C}^3 \times U \to \mathbb{C} \) that satisfy the admissibility condition:

\[ \phi(u, v, w; z, \xi) \in \Omega, \]

\[ \phi(u, v, w; z, \xi) \in \Omega. \]
whenever
\[ u = q(\xi), \quad v = \frac{k\xi q(\xi) + p(1 - \lambda)}{\frac{p}{\lambda}} \]

And
\[
\text{Re} \left\{ \frac{\lambda^2p^2(1 - \lambda)u - 3p^2(1 - \lambda)^2u}{\lambda^2p^2} \right\} \geq k \text{Re} \left\{ \frac{1 + \xi^2q''(\xi)}{q'(\xi)} \right\},
\]
where \( z \in U, \xi \in \partial U \setminus E(q), \lambda > 0 \) and \( k \geq p \).

**Theorem 2.1** Let \( \phi \in \Psi_n[\Omega, q] \). If \( f \in \mathcal{A}(q) \) satisfies
\[
\phi \left( F_{\lambda_p}^m (f \ast g)(z), F_{\lambda_p}^m (f \ast g)(z), F_{\lambda_p}^m (f \ast g)(z) \right) \subset ,
\]
where \( \lambda > 0, m \in N_0 = \{0, 1, 2, \ldots\}, z \in U \).

Then
\[
F_{\lambda_p}^m (f \ast g)(z) < q(z), \quad (z \in U).
\]

**Proof.** By using (1.2) and (1.3), we get the equivalent relation
\[
F_{\lambda_p}^m (f \ast g)(z) = \frac{z \left( F_{\lambda_p}^m (f \ast g)(z) + p(1 - \lambda) F_{\lambda_p}^m (f \ast g)(z) \right)}{\frac{p}{\lambda}}.
\]

Assume that
\[
G(z) = F_{\lambda_p}^m (f \ast g)(z).
\]

Then
\[
F_{\lambda_p}^m (f \ast g)(z) = \frac{zG(z) + p(1 - \lambda)}{\frac{p}{\lambda}} G(z).
\]

Further computation show that
\[
F_{\lambda_p}^m (f \ast g)(z) = \frac{z^2 G(z) + \left( 1 + 2p(1 - \lambda) \right) z G(z) + p^2(1 - \lambda)^2}{\frac{p}{\lambda^2}} G(z).
\]

Define the transformation from \( \mathbb{C}^2 \) to \( \mathbb{C} \) by
\[
u = r, \quad v = \frac{s + p(1 - \lambda)}{\frac{p}{\lambda}}, \quad w = \frac{t + (1 + 2p(1 - \lambda)) s + p^2(1 - \lambda)^2}{\frac{p}{\lambda^2}} r
\]

Let
\[
\psi(u, v, w; z) = \phi(u, v, w, z) = \phi \left( r, \frac{s + p(1 - \lambda)}{\frac{p}{\lambda}}, \frac{t + (1 + 2p(1 - \lambda)) s + p^2(1 - \lambda)^2}{\frac{p}{\lambda^2}} r \right).
\]

The proof shall make use of Lemma 1.1 using equations (2.3), (2.4) and (2.5), we obtain
\[
\psi(G(z), zG(z), z^2 G(z); z) = \phi(F_{\lambda_p}^m (f \ast g)(z), F_{\lambda_p}^m (f \ast g)(z), F_{\lambda_p}^m (f \ast g)(z), z).
\]
Therefore, by making use (2.1) we get
\[
\psi(G(z), zG(z), z^2 G(z); z) \in .
\]
The proof is completed if it can be show that the admissibility condition for \( \phi \in \Psi_n \) is equivalent to the admissibility for \( \psi \) as given in Definition 1.4. Note that
\[
\frac{t}{s} + 1 = \frac{p^2w + 2p^2(1-\lambda)r - 2p^2(1-\lambda)^2u}{\lambda pv - p\lambda(1-\lambda)u},
\] (2.10)
and hence \( \psi \in \Psi_k \Omega, q \). By Lemma 1.1, \( G(z) < q(z) \), or
\[
F_{\lambda,\rho}(f * g)(z) < q(z).
\]

We consider the special situation when \( \Omega \subset \mathbb{C} \) is a simply connected domain. In this case \( \Omega = h(U) \), where \( h \) is a conformal mapping of \( U \) onto \( \Omega \) and the class is written as \( \Psi_n[h, q] \). The following results is an immediate consequence of Theorem 2.1.

**Theorem 2.2** Let \( \phi \in \Psi_n[h, q] \). If \( f \in \mathcal{A}(p) \) satisfies
\[
\phi(F_{\lambda,\rho}^m(f * g)(z), F_{\lambda,\rho}^{m+1}(f * g)(z), F_{\lambda,\rho}^{m+2}(f * g)(z); z) < h(z)
\]
where \( \lambda > 0, m \in N_0, z \in U \).
Then
\[
F_{\lambda,\rho}^m(f * g)(z) < q(z), \quad (z \in U).
\]
The next result is an extension of Theorem 2.2 to the case where the behavior of \( q(z) \) at \( \partial U \) is unknown.

**Corollary 2.1** Let \( \Omega \subset \mathbb{C}, q \) be univalent in \( U \) and \( q(0) = 0 \). Let \( \phi \in \Psi_n[\Omega, q_\rho] \) for some \( \rho \in (0, 1) \), where \( q_\rho(z) = q(\rho z) \). If \( f \in \mathcal{A}(p) \) and satisfies
\[
\phi(F_{\lambda,\rho}^m(f * g)(z), F_{\lambda,\rho}^{m+1}(f * g)(z), F_{\lambda,\rho}^{m+2}(f * g)(z); z) \in \Omega,
\]
where \( \lambda > 0, m \in N_0 \) and \( z \in U \).
Then
\[
F_{\lambda,\rho}^m(f * g)(z) < q(z), \quad (z \in U).
\]
**Proof** From Theorem 2.1 yields \( F_{\lambda,\rho}^m(f * g)(z) < q_\rho(z) \). The result now deduced from \( q_\rho(z) < q(z) \).

**Theorem 2.3** Let \( h \) and \( q \) be univalent in \( U \), with \( q(0) = 0 \) and set \( q_\rho(z) = q(\rho z) \) and \( h_\rho(z) = h(\rho z) \). Let \( \phi \in \mathbb{C}^3 \times U \to \mathbb{C} \) satisfy one of the following conditions:
1. \( \phi \in \Psi_n[h, q_\rho] \), for some \( \rho \in (0, 1) \).
2. There exists \( \rho_0 \in (0, 1) \) such that \( \phi \in \Psi_n[h_\rho, q_\rho] \), for all \( \rho \in (\rho_0, 1) \). If \( f \in \mathcal{A}(p) \) and satisfies (2.11), then
\[
F_{\lambda,\rho}^m(f * g)(z) < q(z).
\]
**Proof** Case (1): By using Theorem 2.1, we obtain \( F_{\lambda,\rho}^m(f * g)(z) < q_\rho(z) \), since \( q_\rho(z) < q(z) \), we deduce
\[
F_{\lambda,\rho}^m(f * g)(z) < q(z).
\]
Case (2): Let \( G(z) = F_{\lambda,\rho}^m(f * g)(z) \) and \( G_\rho(z) = G(\rho z) \).
Then\( (G_\rho(z), zG_\rho'(z), z^2G_\rho''(z); \rho z) = (G(\rho z), zG'(\rho z), z^2G''(\rho z); \rho z) \in h_\rho(U) \). By using Theorem 2.1 and the comment associated with \( \phi(G(z), zG'(z), z^2G''(z); \omega(z)) \in \Omega \), where \( w \) is any mapping \( U \) into \( U \), with \( w(z) = \rho z \), we obtain \( G_\rho(z) = q_\rho(z) \) for \( \rho \in (\rho_0, 1) \). By letting \( \rho \to 1^- \), we get \( G(z) < q(z) \).
Therefore,
\[
F_{\lambda,\rho}^m(f * g)(z) < q(z).
\]
The next result gives the best dominant of the differential subordination (2.11).

**Theorem 2.4** Let \( h \) be univalent in \( U \) and let \( \phi : \mathbb{C}^3 \times U \to \mathbb{C} \). Suppose that the differential equation
\[ \phi(q(z), zq'(z), z^2q''(z); z) = h(z) \]  

(2.12)

has a solution \( q \) with \( q(0) = 0 \) and satisfy one of the following conditions:

1. \( q \in Q_0 \) and \( \phi \in \Psi_n[h, q] \).
2. \( q \) is univalent in \( U \) and \( \phi \in \Psi_n[h, q_{\rho}] \), for some \( \rho \in (0, 1) \).
3. \( q \) is univalent in \( U \) and there exists \( \rho_0 \in (0, 1) \) such that \( \phi \in \Psi_n[h_{\rho_0}, q_{\rho}] \), for all \( \rho \in \rho_0, 1 \). If \( f \in \mathcal{A}(p) \) satisfies (2.11), then

\[ F_{n,p}^m(f * g)(z) < q(z), \]

and \( q \) is the best dominant.

**Proof.** By using Theorem 2.2 and Theorem 2.3, we deduce that \( q \) is a dominant of (2.11). Since \( q \) satisfies (2.12), it is also a solution of (2.11) and therefore \( q \) will be dominated by all dominants of (2.11). Hence, \( q \) is the best dominant of (2.11).

In the particular case \( q(z) = Mz, M > 0 \), and in view of the Definition 1.4, the class of admissible function \( \Psi_n[\Omega, M] \) denoted by \( \Psi_n[\Omega] \) is described below.

**Definition 2.4** Let \( \Omega \) be a set in \( \mathbb{C}, M > 0 \). The class of admissible functions \( \Psi_n[\Omega, M] \) consists of those functions \( \phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C} \) that satisfy the admissibility condition:

\[ \phi \left( M_{e^{i\theta}} + \frac{d}{\lambda} M_{e^{i\theta}} \right) \in \Omega, \]

where \( \lambda > 0, \theta \in R, R(e^{i\theta}) \geq a(d - 1)M \) for all real \( \theta, d \geq 1, z \in U \).

**Corollary 2.2** Let \( \phi \in \Psi_n[\Omega, q] \). If \( f \in \mathcal{A}(p) \) satisfies

\[ \phi(F_{n,p}^m(f * g)(z), F_{n,p}^{m+1}(f * g)(z), F_{n,p}^{m+2}(f * g)(z); z) \in \Omega, \]

where \( \lambda > 0, m \in N_0, z \in U \) and \( M > 0 \). Then

\[ |F_{n,p}^m(f * g)(z)| < M, (z \in U). \]

**Proof.** By using Theorem 2.1 gives

\[ F_{n,p}^m(f * g)(z) < q(z) = Mz \]

\[ F_{n,p}^m(f * g)(z) < q(z) = Mw(z). \]

Hence

\[ |F_{n,p}^m(f * g)(z)| < M, \text{ where } |w(z)| < 1. \]

In the special case \( \Omega = q(U) = \{w : |w| < 1\} \) the class \( \Psi_n[\Omega, M] \) is simply denote by \( \Psi_n[M] \).

**Corollary 2.3** Let \( \phi \in \Psi_n[\Omega, q] \). If \( f \in \mathcal{A}(p) \) satisfies

\[ \phi(F_{n,p}^m(f * g)(z), F_{n,p}^{m+1}(f * g)(z), F_{n,p}^{m+2}(f * g)(z); z) < M, \]

where \( \lambda > 0, m \in N_0, z \in U \) and \( M > 0 \). Then

\[ |F_{n,p}^m(f * g)(z)| < M, (z \in U). \]

**Differential superordination results associated with linear operator**

**Definition 3.1** Let \( \phi \) be a set in \( \mathbb{C} : q \in Q_0 \cap H[\xi], p \neq 0 \). The class of admissible function \( \Psi_n[\Omega, q] \) consists of those function \( \phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C} \) that satisfy the admissibility condition:

\[ \phi(u, v, w ; \xi) \in \Omega, \]

whenever
u = q(z), v = \frac{1}{k} zq'(z) + \frac{p(1-\lambda)}{\lambda} q(z)

and

Re\left\{ \frac{p^2 w + 2p^2 (1-\lambda) v - 3p^2 (1-\lambda)^2 u}{\lambda p v - p (1-\lambda) u} \right\} \leq \frac{1}{k} Re\left\{ 1 - \frac{zq''(z)}{q'(z)} \right\}

where \( z \in U, \xi \in \partial U \setminus E(q), \lambda > 0 \) and \( k \geq p \).

**Theorem 3.1** Let \( \phi \in \Psi_n[\Omega, q] \). If \( f \in \mathcal{A}(p), F_{x,p}^m(f \ast g)(z) \in H_0 \) and

\[
\phi \left( F_{x,p}^m(f \ast g)(z) , F_{x,p}^{m+1}(f \ast g)(z) , F_{x,p}^{m+2}(f \ast g)(z) \right)
\]

is univalent in \( U \), then

\[
\Omega \subset \phi \left( F_{x,p}^m(f \ast g)(z) , F_{x,p}^{m+1}(f \ast g)(z) , F_{x,p}^{m+2}(f \ast g)(z) \right)
\]

(3.1)

where \( \lambda > 0, m \in N_0, z \in U \), implies

\( q(z) \prec F_{x,p}^m(f \ast g)(z), (z \in U) \).

**Proof** From (2.8) and (3.1), we have

\[
\subset \psi(G(z), zG'(z), z^2 G''(z); \ z, \xi), (z \in U).
\]

From (2.6), we see that the admissibility condition for \( \phi \in \Psi_n[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 1.5. Hence and by Lemma 1.2 we get \( q(z) \prec G(z) \).

\( q(z) \prec F_{x,p}^m(f \ast g)(z), (z \in U) \).

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( h(U) \) for some conforming mapping \( h(z) \) onto \( U \). In this case the class \( \Psi_n[h(U), q] \) is written as \( \Psi_n[h, q] \). Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.1.

**Theorem 3.2** Let \( h(z) \) is analytic on \( U \) and \( \phi \in \Psi_n[h, q] \). If \( f \in \mathcal{A}(p), F_{x,p}^m(f \ast g)(z) \in H_0 \) and

\[
\phi \left( F_{x,p}^m(f \ast g)(z) , F_{x,p}^{m+1}(f \ast g)(z) , F_{x,p}^{m+2}(f \ast g)(z) \right)
\]

is univalent in \( U \), then

\[
h(z) \prec \phi \left( F_{x,p}^m(f \ast g)(z) , F_{x,p}^{m+1}(f \ast g)(z) , F_{x,p}^{m+2}(f \ast g)(z) \right),
\]

(3.2)

where \( \lambda > 0, m \in N_0, z \in U \), implies

\( q(z) \prec F_{x,p}^m(f \ast g)(z), (z \in U) \).

**Proof** From (3.2), we get

\[
h(z) = \Omega \subset \left( F_{x,p}^m(f \ast g)(z) , F_{x,p}^{m+1}(f \ast g)(z) , F_{x,p}^{m+2}(f \ast g)(z) \right),
\]

and also by Theorem 3.1, we get

\( q(z) \prec F_{x,p}^m(f \ast g)(z), (z \in U) \).

Theorems 3.1 and 3.2, can only be used to obtain subordinations of differential superordination of the form (3.1) or (3.2). The following Theorem proof the existence of the best subordinate of (3.2) for certain \( \phi \).

**Corollary 3.1** Let \( h_1(z) \) and \( q_1(z) \) be analytic functions in \( U, h_2(z) \) be univalent function in \( U, q_2(z) \in H_0 \) with \( q_1(0) = q_2(0) = 0 \) and \( \phi \in \Psi_n[b_1, q_2] \cap \Psi_n[h_1, q_1] \). If \( f \in \mathcal{A}(p), F_{x,p}^m(f \ast g)(z) \in H_0 \) and

\[
\phi \left( F_{x,p}^m(f \ast g)(z) , F_{x,p}^{m+1}(f \ast g)(z) , F_{x,p}^{m+2}(f \ast g)(z) \right)
\]

is univalent in \( U \), then

Combining Theorem 2.2 and 3.2, we obtain the following sandwich type Theorem.
\[ h_1(z) \prec \phi \left( F_{\lambda, \gamma}^{m_1} (f \ast g)(z), F_{\lambda, \gamma}^{m_2} (f \ast g)(z), F_{\lambda, \gamma}^{m_3} (f \ast g)(z) \right) \prec h_2(z) \]  
when \( \lambda > 0, m \in \mathbb{N}, z \in U \), implies that

\[ q_1(z) \prec F_{\lambda, \gamma}^{m} (f \ast g)(z), \prec q_2(z), \quad (z \in U). \]

References


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