RESEARCH ARTICLE

INTEGRAL SOLUTIONS OF THE NON HOMOGENEOUS HEPTIC EQUATION WITH FIVE UNKNOWNS $x^3-y^3-(x^2+y^2)+z^3-w^3=2+15p^7$

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ABSTRACT

The non-homogeneous Diophantine equation of degree seven with five unknowns represented by $x^3 - y^3 - (x^2 + y^2) + z^3 - w^3 = 2 + 15p^7$ is analyzed for its non-zero distinct integer solutions. A few interesting relation between the solutions and special numbers namely Polygonal numbers, Pyramidal numbers, centered Polygonal numbers are exhibited.

INTRODUCTION

Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity as can be seen from [L.E.Dickson, 1952, L.J.Mordell, 1969, Telang,S.G, 1996, Carmichael, R.D, 1959]. The problem of finding all integer solutions of a diophantine equation with three or more variables and degree at least three, in general presents a good deal of difficulties. There is vast general theory of homogeneous quadratic equations with three variables. Cubic equations in two variables fall into the theory of elliptic curves which is a very developed theory but still an important topic of current research.

A lot is known about equations in two variables in higher degrees. For equations with more than three variables and degree at least three very little is known. It is worth to note that undesirability appears in equations, even perhaps at degree four with fairly small co-efficients. It seems that much work has not been done in solving higher order Diophantine equations. In [Gopalan M.A *et al* 2007, 2010 a,b,c,d,e,f; 2011 a,b,c,d; 2012 a,b,c; 2014 Jayakumar *et al* 2014, Manju somanath *et al* 2011 a,b; 2012, 2014 a, b, 2015, Manjula *et al* 2015] a few higher order equations are considered for integral solutions. In this communication a seventh degree non-homogeneous equation, with five variables represented by $x^3 - y^3 - (x^2 + y^2) + z^3 - w^3 = 2 + 15p^7$ is considered and in particular a few interesting relations among the solutions are presented.

Notations

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

CP_n^m : Centered Pyramidal number of rank n with size m.

J_n : Jacobsthal number of rank n

j_n : Jacobsthal-Lucas number of rank n

ky_n : keynea number of rank n.

Method of Analysis

The non-homogeneous heptic equation with five unknowns to be solved for its distinct non-zero integral solution is

$$x^3 - y^3 - (x^2 + y^2) + z^3 - w^3 = 2 + 15p^7 \qquad (1)$$

Introduction of the linear transformations

$$x = u + 1, y = u - 1, z = v + 1, w = v - 1 \qquad (2)$$

in (1) leads to

$$4u^2 + 6v^2 = 15p^7 \qquad (3)$$

Different methods of obtaining the patterns of integer solution to (1) are illustrated below:

Pattern:1

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Let $p = 4a^2 + 6b^2$ (4)

Using (4) in (3) and applying the method of factorization define,

$$(2u + i\sqrt{6}v) = (3 + i\sqrt{6})(2\alpha + i2\sqrt{6}\beta)$$ (5)

where $(2\alpha + i2\sqrt{6}\beta) = (2a + i\sqrt{6}b)^7$
from which we have,

$$\left.\begin{array}{l}\alpha = 2^6 a^7 - 2016a^5 b^2 + 5040a^3 b^4 - 1512ab^6 \\ \beta = 224a^6 b - 1680a^4 b^3 + 1512a^2 b^5 - 108b^7\end{array}\right\}$$ (6)

Equating real and imaginary parts in (5), we have

$$\left.\begin{array}{l}u = 3\alpha - 6\beta \\ v = 2\alpha + 6\beta\end{array}\right\}$$ (7)

Using (7) in (2), the values of x,y,z and w are given by

$$\left.\begin{array}{l}x(a,b) = 3\alpha - 6\beta + 1 \\ y(a,b) = 3\alpha - 6\beta - 1 \\ z(a,b) = 2\alpha + 6\beta + 1 \\ w(a,b) = 2\alpha + 6\beta - 1\end{array}\right\}$$ (8)

Thus (4) and (8) represent the non-zero distinct integral solutions to (1)

Properties

$(i)2y(n, 1) + z(n, 1) - 4w(n, 1)$
$\qquad = 3[-560t_{4,n}(24F_{4,n,5} + 6F_{4,n,6} - 6CP_n^{13}$
$\qquad - 18t_{3,n} - 32t_{4,n} + 27) - 1079]$

$(ii)x(1,n) + y(1,n) + z(1,n) + w(1,n) - 640$
$\qquad = -5040t_{4,n}[24F_{4,n,5} - 3CP_n^{16} - CP_n^{12}$
$\qquad - 16t_{3,n} - 11t_{4,n} + 4]$

$(iii)2x(n, 1) - 3w(n, 1) - 2597$
$\qquad = -1344t_{4,n}[24F_{4,n,5} + 6F_{4,n,6} - 6CP_n^{11}$
$\qquad - 3CP_n^4 - CP_{12,n} - 35t_{4,n} + 28]$

$(iv)y(1,n) = z(1,n) - 320$
$\qquad = 2520t_{4,n}[3CP_n^{14} + 2CP_n^9 + 6t_{3,n} + 16t_{4,n}$
$\qquad - 24F_{4,n,5}]$

Pattern:2

Consider (3) as

$$4u^2 + 6v^2 = 15p^7 \quad 1$$ (9)

Write 1 as

$$1 = \frac{(5+i2\sqrt{6})(5-i2\sqrt{6})}{49}$$ (10)

Substituting (4) and (10) in (9) and employing the factorization method, define

$$(2u + i\sqrt{6}v) = (3 + i\sqrt{6})(2\alpha + i2\sqrt{6}\beta)\frac{(5+i2\sqrt{6})}{7}$$

Equating real and imaginary parts we have,

$$\left.\begin{array}{l}u = \frac{1}{7}[3\alpha - 66\beta] \\ v = \frac{1}{7}[22\alpha + 6\beta]\end{array}\right\}$$ (11)

As our interest is on finding integer solutions, we have choose α and β suitably so that u and v are integers. Replace a by 7a and b by 7b in (6).

Substituting the corresponding values of α and β in (11) and employing (2), the non-zero integral solutions to (1) are found to be

$x(a,b) = 7^6(3\alpha - 66\beta) + 1$
$y(a,b) = 7^6(3\alpha - 66\beta - 1$
$z(a,b) = 7^6(22\alpha + 6\beta) + 1$
$w(a,b) = 7^6(22\alpha + 6\beta) - 1$
$\quad p(a,b) = 7^6(4a^2 + 6b^2)$

Properties

$(i)x(1,n) + 11z(1,n) - 12$
$\qquad = 7^6 * 1960\{8$
$\qquad - 63t_{4,n}[24F_{4,n,5} - 3CP_n^{16} - 3CP_n^4 - 12t_{3,n}$
$\qquad - 13t_{4,n} + 7]\}$

$(ii)p(2^n, 2^n) - 14^2 j_{2n} - 882J_{2n}$ is an even integer

$(iii)22y(n, 1) - 3w(n, 1) + 19$
$\qquad = 7^8$
$\qquad * 30\{56t_{4,n}[24F_{4,n,7} - 24F_{4,n,3} - 3CP_n^{16}$
$\qquad - t_{10,n} - 22t_{4,n} + 27] - 108\}$

$(iv)x(1,n) + 11w(1.n) + 10$
$\qquad = 245$
$\qquad 7^6\{-504t_{4,n}[24F_{4,n,5} - 6CP_n^{11} + CP_n^6$
$\qquad - 12t_{4,n} - 14t_{3,n} + 4] + 2^6\}$

Pattern:3

In (3), 15 can be written as

$$15 = \frac{(27+i\sqrt{6})(27-i\sqrt{6})}{49}$$ (12)

Using (12), (4) in (3) and applying the method of factorization, define

$$(2u + i\sqrt{6}v) = (2\alpha + i2\sqrt{6}\beta)\frac{(27 + i\sqrt{6})}{7}$$

Equating real and imaginary parts we have

$$u = \frac{1}{7}[27\alpha - 6\beta]$$

$$v = \frac{1}{7}[2\alpha + 54\beta]$$

Repeating the process as in the above pattern:2, the non-zero distinct integral solutions to (1) are obtained as

$x(a,b) = 7^6(27\alpha - 6\beta) + 1$
$y(a,b) = 7^6(27\alpha - 6\beta - 1$
$z(a,b) = 7^6(2\alpha + 54\beta) + 1$
$w(a,b) = 7^6(2\alpha + 54\beta) - 1$
$\quad p(a,b) = 7^2(4a^2 + 6b^2)$

Properties

$(i)9x(1,n) + z(1,n) - 10$
$\qquad = 7^6$
$\qquad * 245\{-504t_{4,n}[24F_{4,n,5} - 3CP_n^{16} - CP_n^{12}$
$\qquad - 16t_{3,n} - 11t_{4,n} + 11] + 64\}$

$(ii)2y(n, 1) - 27w(n, 1) - 25$
$\qquad = 7^6 * 1470\{56t_{4,n}[2t_{4,n}(2t_{4,n_{15}}) + 27]$
$\qquad - 108\}$

$(iii)p(2^{2n}, 2^{n+1}) = 7^2[KY_{n+1} + j_{n+2} + 9J_{2n+3} - 1]$

$$(iv) \, 9x(1,n) + w(1,n) - 8$$
$$= 7^6$$
$$245\{504t_{4,n}[CP_n^{16} + CP_n^4 + 9Pr_n + 10t_{4,n}$$
$$- 24F_{4,n,5} - 4] + 2^6\}$$

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CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous heptic equation with five unknowns. As the heptic equations are rich in variety, one may search for other forms of heptic equation with variables greater than or equal to five and obtain their corresponding properties.

Acknowledgement

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