EXISTENCE OF \((\Phi \otimes \Psi)\) BOUNDED SOLUTIONS FOR FIRST ORDER KRONECKER PRODUCT SYSTEMS ON TIME SCALES

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ABSTRACT

In this paper we unify results recently established for linear differential equations and linear difference systems on time scales and deduce the existing results as particular case.

INTRODUCTION

It is a well recognized fact that mathematical models or equations that describe physical or biological phenomena are in most cases linear differential/difference equations of first order. In recent years the theory of difference equations is gaining attention in many disciplines. In spite of this tendency or inter dependency there is a striking similarity or even duality between the theory of continuous and discrete systems. This paper unifies both continuous and discrete systems in a single frame work by using time scale dynamical systems. In this paper, we shall be concerned with the dynamical systems

\[ x^{\Delta}(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0 \quad (1.1) \]

where \(x(t), f(t)\) are in time scale \(T\) and \(A\) is a continuous/discrete \((k \times k)\) matrix and all scalars are assumed to be real. The calculus of time scales was first initiated by Stefan Hilger [15] in order to create a theory that unify discrete and continuous systems. The use of time scale dynamical systems was first extended to boundary value problems by Murty and Rao in a remarkable paper published in 1994 [12]. Later, more than one hundred and fifty papers came into light during the last 30 years. For more information and recent investigation on time scale dynamical systems, we refer Martin Bohner and Alan Peterson [2]. Existence of \(\Psi\)-bounded solutions for linear system of differential equations are established in [1] and \(\Psi\)-bounded solutions for linear system of equations on timescales were established in a remarkable paper by Kasi Viswanadh et al [6]. Qualitative properties of first order difference systems are taken from [8,10,11]. A new approach to the construction of a transition matrix and application to control system are established in the conference paper [13].

The general idea which was the main goal of Bohner and Peterson was to prove a result for a dynamic equations where the domain of an unknown function is so called time scale or \(\Delta\)differentiable function. If \(T = \mathbb{R}\), the general results obtained yields the same result concerning differential system of first order and \(T = \mathbb{Z}\), the result is the same result concerning a difference system of first order. However there are many other time scales that may work besides the real and the integers, one has a more general result.

The present paper unifies the results of \(\Phi \otimes \Psi\) bounded solutions of linear differential equations established by Kasi Viswanadh et.al [7] and for linear difference equations established by Kasi Viswanadh et. al [5] and by Charyulu L.N. et.al [3]. The results established on initial value problems associated with first order linear difference and differential
equations are taken from [9]. This paper is organized as follows: Section 2, deals with basic concepts on Time Scale dynamical systems and Section 3, presents results so far established on Time scale dynamical systems and our main results are established in section 4.

**Basic results on Time Scale Dynamical Systems**

In this section, we outline some of the basic notions on time scale dynamical systems. A time scale $T$ is a closed subset of $\mathbb{R}$, and examples of time scales include, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, Contour set, Fuzzy sets, Topological sets etc. The set

$$Q = \{ t \in R - Q, 0 \leq t \leq 1 \}$$

are not time scales. Time scales need not necessarily connected. In order to overcome this deficiency, we introduce the notion of Jump operators defined here under:

The operator $\sigma(t) = \text{Inf}\{ t \in T; s > t \}$

$$\rho(t) = \text{Sup}\{ t \in T; s < t \}$$

are called Jump operators. If $\sigma$ is bounded above and $\rho$ is bounded below, then we define

$$\sigma(\max T) = \max T, \sigma(\min T) = \min T$$

A point $t \in T$ is said to be right dense, if $\sigma(t) = t$, right scattered if $\sigma(t) > t$, left dense if $\rho(t) = t$ and left scattered, if $\rho(t) < t$.

The graininess $\mu: [0, \infty) \rightarrow [0, \infty)$ is defined as

$$\mu(t) = \sigma(t) - t.$$ We say that $f$ is rd-Continuous, if it is continuous at right dense points and if

$$\lim_{s \to t} f(s) \text{ as } s \to t \text{ exists at all right dense points } t \in T.$$ A function $f: T \rightarrow \mathbb{R}$ is said to be differentiable at $t \in T^k$ if

$$\lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists and is said to be differentiable for each $t \in T^k$.

A function $f: T \rightarrow \mathbb{R}$ with $F^\lambda(t) = f(t)$ for all $t \in T^k$ is said to integrable, if

$$\int_a^b f^\lambda(c) = F(t) - F(s),$$

where $\lambda$ is the anti-delta derivative of $F$ for all $s, t \in T$.

Let $f: T \rightarrow \mathbb{R}$, $a, b \in T$ then $F^\lambda(t) = f(t)$ and

$$\int_a^b f(t) \Delta t = \int_a^b f(t) dt,$$

and if $T = \mathbb{Z}$, then $F^\lambda(t) = \Delta f(t) = f(t + 1) - f(t)$ and

$$\int_a^b f(t) \Delta t = \begin{cases} \sum_{k=0}^{b-1} f(k), & \text{if } a < b \\ 0, & \text{if } a = b \\ \sum_{k=a}^{b-1} f(k), & \text{if } a > b \end{cases}$$

Note that if $f$ is $\Delta$-differentiable, then $f$ is continuous. Further if it is right scattered and $f$ is continuous at $t$, then

$$f^\lambda(t) = \frac{f(\sigma(t))-f(s)}{\mu(t)},$$

for a complete survey on $\Delta$- differentiable functions, we refer to an excellent survey made by M. Bhoner and Allan Peterson [].

Definition 2.1: A matrix $P$ is said to be a projection matrix, if $P^2 = P$. If $P$ is the projection matrix, then $I - P$ is also a projection. Two such projections whose sum is $I$ and whose product is zero, are said to be complementary.

For any $(k \times k)$ real matrix $A = (a_{ij})$, $i, j = 1, 2, \ldots, k$, we define the norm of the matrix $A$ as

$$\|A\| = \sup_{x \in \mathbb{R}^k} \|Ax\| = \|A\|.$$

Definition 2.2: A mapping $f: T \rightarrow X$, where $X$ is a Banach space is called rd-continuous if (1) if it is continuous at each right dense point $t \in T$. If $t$ is right scattered and $f$ is continuous at $t$, then

$$f^\lambda(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)},$$

where $\mu(t) = \sigma(t) - t$.

Definition 2.3: Any set of $n$-linearly independent solutions $y_1, y_2, \ldots, y_n$ of

$$y^\lambda(t) = A(t)y(t)$$

is called a fundamental set and the matrix with $y_1, y_2, \ldots, y_n$ is its columns is called a fundamental matrix and is denoted by $Y$.

**On Existence of $\Psi$ - bounded solutions**

In recent years the concept of $\Psi$ bounded solutions of linear differential/difference equations of first order gained momentum in the theory of differential equations/difference equations. We establish the general solution of the Kronecker product of first order systems in terms of two fundamental matrices of the systems

$$x^\lambda(t) = A(t)x(t) \quad (3.1)$$

and

$$y^\lambda(t) = B(t)y(t) \quad (3.2)$$

where $A$ and $B$ square matrices of orders $m \times m$ and $n \times n$ respectively and $x$ and $y$ are appropriate column vectors and $\Delta$ stands for first order derivative. $(3.1)$ and $(3.2)$ are embedded in a first order Kronecker product system as

$$[x(t) \otimes y(t)]^\lambda = [x^\lambda(t) \otimes y(t) + x(t) \otimes y^\lambda(t)]$$

$$= A(t)x(t) \otimes y(t) + x(t) \otimes B(t)y(t)$$

$$= \left[(A(t) \otimes I_m) + (I_m \otimes B(t))\right][x(t) \otimes y(t)]$$

where $D = \left[(A(t) \otimes I_m) + (I_m \otimes B(t))\right]$.

In case $T = \mathbb{Z}$, we have

$$x(n+1) \otimes y(n+1) = A(n)x(n) \otimes B(n)y(n)$$

$$= [(A(n) \otimes I_m) + (B(n) \otimes I_m)][x(n) \otimes y(n)] \quad (3.3)$$

We shall now establish the general solution of the Kronecker product time scale dynamical systems in terms of the two fundamental matrices.

Consider the linear $\Delta$- differentiable system

$$x^\lambda(t) = A(t)x(t) + b(t), \quad x(t_0) = x_0$$

(3.4)
where \( t \in T \). Any solution of the \( \Delta \)-differential system is given by
\[
x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \sigma(\zeta))b(\zeta) d\zeta
\]
(3.5)
Now when \( t \in R \) then (3.4) reduces
\[
x'(t) = A(t)x(t) + b(t), \quad x(t_0) = x_0
\]
Any solution \( x(t) \) is given by
\[
x(t) = \Phi(t, t_0)x_0 + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(\zeta)b(\zeta) d\zeta
\]
(3.6)
where \( \Phi \) is a \((k \times k)\) matrix and \( B \) is a \((p \times p)\) matrix, then
\[
(x \otimes y)^\Delta = \left[ (A(t) \otimes I_p) + (l_k \otimes B(t)) \right] (x \otimes y)
\]
(3.7)
where \( A \) is a \((k \times k)\) matrix and \( B \) is a \((p \times p)\) matrix, then
\[
(x \otimes y)' = \left[ (A(t) \otimes I_p) + (l_k \otimes B(t)) \right] (x \otimes y)
\]
(3.8)
when \( t \in R \) then
\[
(x \otimes y)' = \left[ (A(t) \otimes I_p) + (l_k \otimes B(t)) \right] (x \otimes y)
\]
(3.9)
when \( t \in N \) then
\[
(x \otimes y)(n+1) = \left[ (A(n) \otimes I_p) + (l_q \otimes B(t)) \right] (x \otimes y)
\]
(3.10)
Theorem 3.1: Any solution of the Kronecker product system (2.7) satisfying
\[
(x \otimes y)(n_0) = (C_1 \otimes C_2) \text{ is given by } (x_n \otimes y_n) = \Phi(n, n_0) \otimes \Psi(n, n_0) (C_1 \otimes C_2),
\]
where \( \Phi \) and \( \Psi \) are the fundamental matrices of the system (3.1) and (3.2). Let \( Y(t) \)
and \( Z(t) \) be fundamental matrices of (3.1) and (3.2) respectively. Then we have the following theorem:
\[
[Y(t) \otimes Z(t)]^\Delta = [Y^\Delta(t) \otimes Z(t) + Y(t) \otimes Z^\Delta(t)]
\]
\[
= [(A(t) \otimes I_p) + (l_q \otimes B(t))] [Y(t) \otimes Z(t)]
\]
Therefore \( Y(t) \otimes Z(t) \) is a fundamental matrix of (3.4) Conversely, suppose \( Y(t) \otimes Z(t) \) is a fundamental matrix of (3.4). Then it is claimed that \( Y(t) \) and \( Z(t) \) are fundamental matrix solutions of (3.1) and (3.2). For
\[
[Y(t) \otimes Z(t)]^\Delta = [Y(t) \otimes Z(t)] + [Y(t) \otimes B(t)Z(t)]
\]
Now
\[
[Y^\Delta(t) \otimes Z(t)] + [Y(t) \otimes Z^\Delta(t)]
\]
\[
= [(A(t) \otimes I_p) + (l_q \otimes B(t))] [Y(t) \otimes Z(t)]
\]
Therefore
\[
(Y^\Delta(t) - A(t)Y(t)) \otimes Z(t) = Y(t) \otimes (B(t)Z(t) - Z^\Delta(t))
\]
Multiplying both sides with \( Y^{-1} \otimes Z^{-1}(t) \), we get the above equality holds, if each ratio is either a zero matrix or a unit matrix. If each ratio is a zero matrix, then
\[
Y^\Delta(t) - A(t)Y(t) = 0
\]
Or \( Y \) is a fundamental matrix of \( x^\Delta(t) = A(t)x(t) \) and \( Z \) is a fundamental matrix of
\[
y^\Delta(t) = B(t)y(t)
\]
if each ratio is a unit matrix then
\[
Y^\Delta(t) = (I + A(t))Y(t)
\]
and hence a contradiction. Similar contradictions arises in the other scenario.

**Main Results**

In this section, we shall be concerned with the main question of existence of \( \Phi \otimes \Psi \) bounded solution of the Kronecker Product System (3.8). We assume that the system (Homogeneous) admits at least one \( \Phi \otimes \Psi \) bounded solution for every Lebesgue \( \Phi \otimes \Psi \)-delta integrable functions. Further we suppose that the system (3.6) admits a
\[
\Phi \text{-bounded solution for every Lebesgue } \Phi \text{-delta integrable function and (3.7) admits a}
\]
\[\Psi\text{bounded solution for every Lebesgue } \Psi \text{-delta integrable functions. Let the vector space } R \text{ be represented as a directsum of the three subspaces } X_1, X_2, X_3 \text{ such that a solution } y \text{ of (3.6) is } \Phi \text{-bounded solution } x(0) \in X_0 \text{ and } y \text{ bounded on } T^* = [0, \infty] \text{ if and only if } x(0) \in X_0 \Rightarrow X_0. \text{ Similarly every solution } y \text{ of (2.4) is } \Psi \text{-bounded on } T \text{ if and only if } y(0) \in X_0 \text{ and } \Psi \text{ bounded on } T^* \text{ if and only if } y(0) \in X_0 \Rightarrow X_0. \text{ Also we assume that } P_n, P_0, P_\delta \text{ denote the corresponding projections on } X_1, X_0, X_3 \text{ respectively. To simply our results further we use the following notations.}
\]
\[\alpha(t) = (\Phi \otimes \Psi)(t) \text{ and } \beta(t) = (Y \otimes Z)(t) \text{ and } \chi(t) = (x \otimes y)\]
where \( \alpha \) and \( \beta \) are square matrices of order \( kp \times kp \).
Theorem 4.1: If A and B are continuous \((k \times k)\) and \((p \times p)\) matrices, then (3.8) has at least one \(\Phi \otimes \Psi = \alpha\) bounded solution on \(T\) for every Lebesgue \(\beta\)-delta integrable function \(f: T \to \mathbb{R}\) and only if there exists a positive constant \(K > 0\) such that

\[
\|\alpha(t)\beta(t)P_{-1}\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K \quad \text{for} \quad t > 0, \sigma(s) \leq 0.
\]

\[
\|\alpha(t)\beta(t)(P_{0} + P_{\beta})\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K \quad \text{for} \quad t > 0, \sigma(s) < 1
\]

\[
\|\alpha(t)\beta(t)P_{0}\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K \quad \text{for} \quad t > 0, \sigma(s) > 0, \sigma(s) > t
\]

\[
\|\alpha(t)\beta(t)P_{0}\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K \quad \text{for} \quad t \leq 0, \sigma(s) > t
\]

\[
\|\alpha(t)\beta(t)(P_{0} + P_{\beta})\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K \quad \text{for} \quad t \leq 0, \sigma(s) < 0, \sigma(s) < 0
\]

\[
\|\alpha(t)\beta(t)P_{0}\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\| \leq K \quad \text{for} \quad t \leq 0, \sigma(s) \geq t, \sigma(s) \geq 0
\]

Proof: First suppose that the system (3.8) has at least one \(\alpha\)-bounded solution on \(T\) for every Lebesgue \(\alpha - \Delta\) integrable function \(f: T \to \mathbb{R}\). Then we define

\[ C_{\alpha} \]: The Banach space of all \(\alpha\)-bounded continuous functions \(x: T \to T\) with norm

\[ \|x\|_{C_{\alpha}} = \text{Sup}_{t \in T} \|\alpha(t)x(t)\|, \]

\[ \mathcal{B} : \text{The Banach space of all } \alpha - \Delta \text{ integrable functions } x: T \to T \text{ with norm} \]

\[ \|x\|_{\mathcal{B}} = \int_{-\infty}^{\infty} \|\alpha(t)x(t)\| \Delta t \]

\[ \mathcal{D} : \text{The set of all functions } x: T \to T \text{ which are absolutely continuous functions on all real intervals } J \subset T \alpha \text{-bounded on } T, x(0) \in X \otimes X \text{ and } (x \otimes y)\Delta (t) = \alpha\Delta (t) = [A(t) \otimes I_{p}] + [I_{p} \otimes B(t)]\alpha(t) \in \mathcal{B} \]

Note that \(\mathcal{D}\) is a vector space and

\[ x \to \|x\|_{\mathcal{B}} = \|x\|_{C_{\alpha}} + \alpha - [A(t) \otimes I_{p}] + [I_{p} \otimes B(t)]\alpha \text{is a norm on } \mathcal{D}. \]

Step 1: (\(\mathcal{D}, \|\cdot\|_{\mathcal{D}}\)) is a Banach space. For, let \(\{x_{n}\}, n \in \mathbb{N}\) be a fundamental sequence in \(C_{\alpha}\). Therefore, there exists a continuous \(\alpha\)-bounded solution \(\hat{x}: T \to T\) such that

\[ \lim_{n \to \infty} \|x_{n}(t) - \hat{x}(t)\| = 0 \quad \text{uniformly on} \quad \text{every compact subset of } T. \]

Thus \(\hat{x}(0) \in X \otimes X\), where \(\hat{x} = (x \otimes y)\Delta t\).

Step 2: There exists a positive constant \(K > 0\) such that for every \(b \in \mathcal{B}\) and for every corresponding solution \(\hat{x} \in \mathcal{D}\) of (2.5), we have

\[ \sup_{t \in \mathcal{T}} \|\alpha(t)\hat{x}(t)\| \leq K \int_{-\infty}^{\infty} \|\alpha(t)f(t)\| \Delta t. \]

For, define the mapping \(T: \mathcal{D} \to \mathcal{B}\) such that

\[ T\hat{x} = [\hat{x} - (A \otimes I_{p})] \hat{x}. \]

Let \(f \in \mathcal{B}\) and let \(\hat{x}\) be the \(\alpha\) bounded solution of \(T\) of the system (2.5).

Let \(\hat{x}\) be the solution of the Cauchy problem

\[ \hat{x}(t) = \left[([A(t) \otimes I_{p}]) + ([I_{p} \otimes B(t)])\right] \hat{x}(t), \]

satisfying \(\hat{x}(0) = \hat{x}(0) = \hat{x}(0) = P_{0} \hat{x}(0) = 0\.

Then \(u(t) = \hat{x}(t) - \tilde{x}(t)\) is a solution of (3.8) with \(u(0) = \hat{x}(0) = \hat{x}(0) = P_{0} \hat{x}(0) = 0\.

From the definition of \(X_{0}\), it follows

that \(\tilde{x}(t)\) is \(\alpha\)-bounded on \(T\). Thus \(\tilde{x}\) is bounded on \(T\). Thus \(\tilde{x}\) belongs to \(\mathcal{D}\) and \(T\tilde{x} = f\). Consequently \(T\tilde{x}\) is onto. From a fundamental theorem on Banach space, it follows that \(T^{-1}\) is also bounded (this is a fact of sense of the closed mapping theorem in Banach spaces and can be found in Simmons [14].)

Thus \(\|T^{-1}\|\|f\|_{\mathcal{B}} \leq \|T\|\|\|f\|_{\mathcal{B}}\) for all \(f \in \mathcal{B}\). Thus step 2 follows.

Step 3: Let \(\theta_{1} < \theta_{2} < \theta_{3}\) be fixed points but arbitrary and let \(f: T \to T\) be a function in \(\mathcal{B}\) which vanishes on \((-\infty, \theta_{1}) \cup \theta_{2} \cup \theta_{3}\).

It can easily be verified that the functions \(x: T \to T\) defined by

\[ x(t) = \int_{\theta_{1}}^{t} \alpha(t)\beta(t)(P_{0} + P_{\beta})\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\Delta s, \theta_{1} \leq t < \theta_{2}, \]

\[ = \int_{\theta_{2}}^{\theta_{3}} \alpha(t)\beta(t)(P_{0} + P_{\beta})\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\Delta s, \theta_{2} \leq t < \theta_{3}, \]

\[ = \int_{\theta_{3}}^{t} \alpha(t)\beta(t)(P_{0} + P_{\beta})\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\Delta s, \theta_{3} \leq t < \theta_{4}, \]

is the solution of the Kronecker Product system (2.5).

Now, we take

\[ G(t, \sigma(s)) = \begin{cases} \alpha(t)P_{0}\alpha^{-1}(\sigma(s)), & \sigma(s) \leq 0 < t \leq 0 \\ \alpha(t)(P_{0} + P_{\beta})\alpha^{-1}(\sigma(s)), & 0 < \sigma(s) < t \leq 0 \end{cases} \]

\[ = -\alpha(t)(P_{0} + P_{\beta})\alpha^{-1}(\sigma(s)), & 0 < \sigma(s) < t < 0 \]

\[ \alpha(t)P_{0}\alpha^{-1}(\sigma(s)), & \sigma(s) \geq t \leq 0 \]

\[ \text{is continuous on } T \times T \text{ except on the time } t = \sigma(s), \text{ where it has a Jump discontinuities. Thus, we have} \]

\[ \int_{0}^{\theta_{1}} \int_{0}^{\theta_{1}} \alpha(t)\beta(t)(P_{0} + P_{\beta})\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\Delta s = \int_{0}^{\theta_{2}} \alpha(t)\beta(t)(P_{0} + P_{\beta})\beta^{-1}(\sigma(s))\alpha^{-1}(\sigma(s))\Delta s \]

\[ = -\int_{0}^{\theta_{2}} \alpha(t)P_{0}\alpha^{-1}(\sigma(s))\Delta s \]

\[ = \int_{0}^{\theta_{2}} \alpha(t)P_{0}\alpha^{-1}(\sigma(s))\Delta s = \alpha(t) \]

For \(t \in [\theta_{1}, 0)\), we can show that
\[\int_{0}^{\delta} \bar{G}(t, \sigma(s)) f(s) \Delta s = -\int_{0}^{1} \alpha(t)(P_{0} + P_{s}) \omega^{-1}(\sigma(s)) f(s) \Delta s - \int_{t}^{1} \alpha(t)(P_{0} + P_{s}) \omega^{-1}(\sigma(s)) f(s) \Delta s - \int_{0}^{\delta} \alpha(t) P_{s} \omega^{-1}(\sigma(s)) f(s) \Delta s = -\int_{0}^{1} \alpha(t) P_{s} \omega^{-1}(\sigma(s)) f(s) \Delta s + \int_{0}^{\delta} \alpha(t) P_{s} \omega^{-1}(\sigma(s)) f(s) \Delta s = x(t) \]

Similarly, in other cases, we can show that each integral is equal to \(x(t)\). Hence
\[
\|\alpha(t) G(t, \sigma(s)) \omega^{-1}(\sigma(s))\| \leq K
\]

Now, to prove the other parts, suppose that \((Y \otimes Z)\) is a fundamental matrix of the Kronecker product system (3.4). Then by theorem (3.1), \(Y\) is fundamental matrix of the system (2.3) and \(Z\) is fundamental matrix of the (2.4). Let \(f: T \rightarrow T\) be a Lebesgue \(\alpha\)-delta integrable function on \(T\); we consider the function \(u: T \rightarrow T\) as
\[
u(t) = \int_{-\infty}^{1} \alpha(t)(P_{0} + P_{s}) \omega^{-1}(\sigma(s)) f(s) \Delta s + \int_{0}^{\delta} \alpha(t) P_{s} \omega^{-1}(\sigma(s)) f(s) \Delta s = x(t) \]

Now, the function is well defined on \(T\). Indeed, for \(\nu < t \leq 0\), we have
\[
\int_{\nu}^{\delta} \|\alpha^{-1}(s) f(s)\| \Delta s = \int_{0}^{\delta} \|\alpha^{-1}(s) f(s)\| \Delta s \leq \|\alpha^{-1}(s)\| \int_{0}^{\delta} \|f(s)\| \Delta s \leq K \|\alpha^{-1}(s)\| \int_{0}^{\delta} \|f(s)\| \Delta s
\]

which shows that the left hand side of the integral is absolutely convergent. For \(t > 0\), we have the similar proof.

It can easily be proved that the function \(u\) is a solution of (3.4) and the solution \(u\) is \(\alpha\)- bounded. The proof of the theorem is complete.

**Theorem 4.2:** If the homogeneous Kronecker product system (3.4) has a non-trivial \(\alpha\)-bounded solution on \(T\), for every Lebesgue \(\alpha\)-delta integrable function \(f: T \rightarrow T\) if and only if there exists a positive constant \(K\) such that
\[
\|\alpha(t) f(t) P_{s} \omega^{-1}(\sigma(s)) \omega^{-1}(\sigma(s))\| \leq K \|\sigma(s)\|
\]

\(-\infty < \sigma(s) < \infty, \|\alpha(t) f(t) P_{s} \omega^{-1}(\sigma(s)) \omega^{-1}(\sigma(s))\| \leq K t \leq \sigma(s)\) for \(-\infty < t \leq \sigma(s) < \infty

Proof: In the proof of theorem 4.1, take \(P_{0} = 0\)

**Theorem 4.3:** A fundamental matrix \(\beta(t) = (Y(t) \otimes Z(t))\) satisfies the following conditions
\[
\lim_{t \to \infty} \|\alpha(t) \beta(t) P_{0}\| = 0
\]
\[
\lim_{t \to \infty} \|\alpha(t) \beta(t) P_{0}\| = 0
\]
\[
\lim_{t \to \infty} \|\alpha(t) \beta(t) P_{0}\| = 0
\]

Note that if \(\|Y(t)\|\) or \(\|Z(t)\|\) goes to zero as \(t \to \infty\) then the result follows.

Note that the results of Kasi Viswanadh. *et al* [7] on \(\Phi \otimes \Psi\)-bounded solutions associated with first order fuzzy linear system of differential equations needs special attention. Also \(\Psi^{a}\)-bounded solutions were established by Kasi Viswanadhin [4]. These results can be extended to time scale dynamical systems and work in this direction is in progress.

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