INITIAL-POINT BOUNDARY VALUE PROBLEMS FOR TLP AND SECOND ORDER FIBONACCI NUMBERS

1Bahram E Çekim and 2*Divya L Nethi

1Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey.
2* Vasudeva Residency, Malavya Nagar, Gudur, A.P, India

ABSTRACT

In this paper, we present a criteria for the existence of $Ψ$-bounded solutions for linear fuzzy first order systems on time scales. The calculus of timescales is used as a tool to unify both continuous and discrete fuzzy systems in a single framework.

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Results 2.1: Let $A$ be an $n \times n$ constant matrix with 

$$|λI - A| = λ^n + c_{n-1}λ^{n-1} + ... + c_1λ + c_0$$

Then the matrix $Φ(t) = e^{At}$ is the unique solution of the nth order scalar differential equation

$$x^{(n)} + c_{n-1}x^{(n-1)} + ... + c_1x' + c_0x = 0$$

satisfying the initial condition $x(0) = 0, x'(0) = A_x, ..., x^{(n-1)}(0) = A^{n-1}_x$.

Result 2.2: Let $A$ be an $n \times n$ constant matrix with

$$|λI - A| = λ^n + c_{n-1}λ^{n-1} + ... + c_1λ + c_0$$

Then

$$e^{At} = x_1(t)I + x_2(t)A + x_3(t)A^2 + ... + x_n(t)A^{n-1}$$

where $x_1, x_2, ..., x_n$ are $n$ linearly independent solutions of the scalar differential equation

$$x^{(n)} + c_{n-1}x^{(n-1)} + ... + c_1x' + c_0x = 0$$

satisfying the initial conditions,

$$x_1(0) = 1, x_2(0) = 0, ..., x_n(0) = 0$$

$$x'_1(0) = 0, x'_2(0) = 1, ..., x'_n(0) = 0$$

$$..., x_{(n-1)}(0) = 0, x_{(n-1)}(0) = 1.$$
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The fact that any scalar differential equation of order n can be recast in the form of a first order differential system. We confine our attention to Kasi Viswanadh V Kanuri et.al in [2] for metrics used and to Kasi Viswanadh V. Kanuri in [11] for applications on time scales to fuzzy dynamical systems

\[ L(y) = y^{(n)} + p_1 y^{(n-1)} + \ldots + p_n y = f(t) \]  

(2.2)

where \( p_1, p_2, \ldots, p_n \) and \( f \) are all \((n+1)\) functions defined on \([a, b]\).

The companion vector equation is formed with,

\[ y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \]

\[ y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \\ y_n' \end{bmatrix} \]

\[ y'' = \begin{bmatrix} y_1'' \\ y_2'' \\ \vdots \\ y_{n-1}'' \\ y_n'' \end{bmatrix} \]

\[ y^{(n)} = \begin{bmatrix} y_1^{(n)} \\ y_2^{(n)} \\ \vdots \\ y_{n-1}^{(n)} \\ y_n^{(n)} \end{bmatrix} \]

Thus \( y_i \) is a solution of the homogeneous equation corresponding to (2.2)

We now proceed to the main results.

**Main Results**

In this section, we establish our main results. Recall that for a first order linear differential equation

\[ y' + p(t)y = g(t) \quad y(t_0) = y_0 \]

if \( p(t) \) and \( g(t) \) are continuous on \([a, b]\), then there exists a unique solution on the interval \([a, b]\).

We can ask the same questions of second order linear differential equations. We need to first make a few comments by referring to KasiViswanadh V Kanuri et.al in [1,3,4]. The first is that for a second order differential equation, it is not enough to state the initial position. We must also have the initial velocity. One way of convincing yourself, is that since we need to reverse two derivatives, two constants of integration will be introduced, hence two pieces of information must be found to determine the constants.

A second comment is that of notation. Let

\[ y'' + p(t)y' + q(t)y = g(t) \]

be a second order linear differential equation. Then we call the operator

\[ L(y) = y'' + p(t)y' + q(t)y \]

the corresponding linear operator. Thus we want to find solutions to the equation

\[ L(y) = g(t) \quad y(t_0) = y_0 \quad y'(t_0) = y'_0 \]

We will state the following theorem without proof. The proof is well above the level of this course.

**Theorem: Existence and Uniqueness**

Let \( p(t) \), \( q(t) \), and \( g(t) \) be continuous on \([a, b]\), then the differential equation

\[ y'' + p(t)y' + q(t)y = g(t) \quad y(t_0) = y_0 \quad y'(t_0) = y'_0 \]

has a unique solution defined for all \( t \) in \([a, b]\).

**Solution**

We first put it into standard form

\[ y'' + 3t/(t^2 - 1)y' + (\cos t)/(t^2 - 1) y = et/(t^2 - 1) \quad y(0) = 4, \quad y'(0) = 5 \]

\( p, q, \) and \( g \) are all continuous except at \( t = -1 \) and \( t = 1 \). The theorem tells us that there is a unique solution on \([-1,1]\).

**Homogeneous Linear Second Order Differential Equations**

Next we will investigate solutions to homogeneous differential equations. Consider the homogeneous linear differential equation

\[ L(y) = 0 \]

We have the following theorem

**Theorem**

Let \( L(y) = 0 \) be a homogeneous linear second order differential equation and let \( y_1 \) and \( y_2 \) be two solutions. Then \( c_1 y_1 + c_2 y_2 \) is also a solution for any pair or constants \( c_1 \) and \( c_2 \).

Using the terminology of linear algebra, we know that \( L \) is a linear transformation of the vector space of differentiable functions into itself. The theorem reminds us that the kernel of a linear transformation is a vector subspace.

**Proof**

\[ L(c_1 y_1 + c_2 y_2) = (c_1 y_1 + c_2 y_2)' + p(t)(c_1 y_1 + c_2 y_2)' + q(t)(c_1 y_1 + c_2 y_2) = c_1 y_1' + c_2 y_2' + p(t) c_1 y_1' + p(t) c_2 y_2' + q(t) c_1 y_1 + q(t) c_2 y_2 = c_1 y_1' + p(t) c_1 y_1' + q(t) c_1 y_1 + q(t) c_2 y_2 = c_1 (y_1' + p(t) y_1') + q(t) (y_1 + q(t) y_2) = c_1 (y_1' + p(t) y_1') + q(t) (y_2') = c_1 L(y_1) + c_2 L(y_2) = 0 + 0 = 0 \]

Next, we investigate the initial conditions. If we find a general solution to the homogeneous system, can we choose constants such that the solution satisfies the initial conditions? That is can we find \( c_1 \) and \( c_2 \) such that

\[ c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \]
\[ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y'_0 \]

We can put this into a matrix equation

\[
\begin{bmatrix}
\hat{y}_1(t_0) \\
\hat{y}_2(t_0) \\
\end{bmatrix} =
\begin{bmatrix}
\hat{y}_1(t_0) \\
\hat{y}_2(t_0) \\
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\end{bmatrix}
\]
This has a unique solution if and only if the determinant of the matrix is not zero. This determinant is called the Wronskian. This proves the following theorem

**Theorem**

Let \( L(y) = 0; y(0) = 0 \) and \( y'(0) = y'0 \) be a homogeneous linear second order differential equation and let \( y1 \) and \( y2 \) be two general solutions (No initial value). Then if the equation \( y1y2' - y1' y2 \) is nonzero, there exists a solution to the initial value problem of the form

\[
y = c1y1 + c2y2
\]

Consider the differential equation

\[
y'' + 2y' - 8y = 0
\]

It is easy to check that the general solution is given by

\[
y = c1e2t + c2e-4t
\]

The Wronskian of

\[
y1 = e2t \quad y2 = e-4t
\]

is given by

\[
e2t(-4e-4t) - (2e2t)e-4t = -6e-2t - 6e-2t
\]

Which is never zero. We can conclude that any initial value problem will have a unique solution of the form

\[
y = c1e2t + c2e-4t
\]

It is a well known fact that many Physical and Technological phenomena can be modeled by kroener product of matrices of the form

\[
(y \otimes z)(t) = (A \otimes I_n + I_n \otimes B)(y \otimes z)(t)
\]

with some initial data \((y \otimes z)(t_0) = (y_0 \otimes z_0)\).

Since the initial data from all types of measurements may have errors, it is reasonable to investigate the extent to which the small changes in the initial data affect the desired behavior of solutions. If a small change in the initial data causes a substantial deviation in the desired behavior solution \((4.1)\), then such a solution is not acceptable even approximately. As such, we investigate the stability of \((4.1)\) in the next theorem.

**Theorem 4.1:** The kroener product of the homogeneous system \((4.1)\) is stable, if and only if there exists a positive constant \(M\) such that

\[
\|K(u)(t)\| \leq M \quad \text{and} \quad \|K(v)(t)\| \leq M
\]

for all \(t \geq t_0\), and asymptotically stable, if and only if either

\[
K(u)(t) \leq M \quad \text{for all} \quad t \geq t_0 \quad \text{and} \quad K(v)(t) \to 0 \quad \text{as} \quad t \to \infty
\]

or

\[
K(v)(t) \leq M \quad \text{for all} \quad t \geq t_0 \quad \text{and} \quad K(u)(t) \to 0 \quad \text{as} \quad t \to \infty.
\]

**Proof:** Suppose \(\|K(u)(t)\| \leq M\) and \(\|K(v)(t)\| \leq M\) for all \(t \geq t_0\). Then any solution of \((4.1)\) is of the form

\[
(y \otimes z)(t) = [K(u)(t) \otimes K(v)(t)](y \otimes z)
\]

\[
\|y \otimes z)(t)\| \leq \|K(u)(t)\| \|K(v)(t)\| \|y\| \|z\|
\]

\[
\leq M^2 \|y\| \|z\|
\]

Choose \(\|y\| \leq M\) and \(\|z\| \leq M\). Then \(\|y \otimes z)(t)\| \leq \varepsilon\) for all \(t \geq t_0\).

Hence \((4.1)\) is stable.

Suppose, the kroener product system is stable. Then it can easily proved that \(\|K(u)(t)\| \leq M\) and \(\|K(v)(t)\| \leq M\) for all \(t \geq t_0\). Further if \(\|K(u)(t)\| \leq M\) and \(\|K(v)(t)\| \to 0 \quad \text{as} \quad t \to \infty\), then \(\|y \otimes z)(t)\| \to 0 \quad \text{as} \quad t \to \infty\) and hence the system is asymptotically stable. Similarly, when \(\|K(u)(t)\| \to 0 \quad \text{as} \quad t \to \infty\) and \(\|K(v)(t)\| \leq M\) for all \(t \geq t_0\) it follows that \(\|y \otimes z)(t)\| \to 0 \quad \text{as} \quad t \to \infty\).

Now, we shall be concerned with the general first order kroener product system

\[
(y \otimes z)(t) = (A \otimes I_n + I_n \otimes B)(y \otimes z)(t) + (D_1 \otimes D_2)(t)(u_1 \otimes u_2)(t)
\]

\[(4.2)\]

\[
(x_1 \otimes x_2)(t) = (C_1 \otimes C_2)(t)(y \otimes z)(t)
\]

\[(4.3)\]

where \(A\) is \((n \times n)\), \(B\) is \((m \times m)\), \(D_1\) is \((n \times p)\), \(D_2\) is \((m \times q)\) \(u_1\) being \((p \times m)\) and \(u_2\) being \((q \times n)\) matrices. \(C_1\) is \((p \times n)\) and \(C_2\) is \((q \times m)\) matrices, \(x_1\) and \(x_2\) being \((p \times 1)\) and \((q \times 1)\) vectors, \(y\) being an \((n \times 1)\) vector and \(z\) being an \((m \times 1)\) vector.

**Definition 4.1:** The linear system \((4.2)\) is completely controllable on \([0, t]\) for any initial time \(t_0\) and initial state \((y \otimes z)(t_0) = (y_0 \otimes z_0)\) if there exist a continuous input signal \((u_1 \otimes u_2)(t)\) such that the corresponding solution of \((4.1)\) satisfies \((y \otimes z)(t_1) = (y_1 \otimes z_1)\).

The fundamental concept of controllability and observability for an mn-input, np-output of an multidimensional kroener product linear state equation \((4.2)\) and \((4.3)\) will be considered in this section. For a time varying kroener product linear state equation, the connection of the input signal to the state variables can change with time. Therefore the concept of controllability is tied up to a specific finite time interval \([0, t_f]\) of course \(t_0 \leq t < t_f\). Hereafter we use the following notation

\[
K(u)(t_0, t) = \Phi(t_0, t)
\]

\[
K(v)(t_0, t) = \Psi(t_0, t)
\]

**Definition 4.2:** The kroener product linear state equation \((4.2)\) is said to be controllable on \([0, t_f]\) if for any given
initial time \((y \otimes z)(t_0) = (y_0 \otimes z_0)\), there exist a continuous input signal \((u_1 \otimes u_2)(t)\) such that the corresponding solution of \((4.1)\) satisfies 
\[(y \otimes z)(t_f) = 0.\]

Note that the kronecker product linear system is completely controllable if it is controllable for all \(t \in [0, t_f]\).

\[\textbf{Theorem 4.2:} \quad \text{The kronecker product state equation (4.2) is completely controllable on the closed interval } J = [0, t_f] \text{ if and only if the } (m \times m) \text{ symmetric matrix}
\]
\[
\begin{bmatrix}
\omega_1(t_{0},t_f) & \omega_2(t_{0},t_f) \\
\omega^T_1(t_{0},t_f) & \omega^T_2(t_{0},t_f)
\end{bmatrix}
\]
is non-singular. \((\omega_1)\) is an \((m \times m)\) and \((\omega_2)\) is \((n \times n)\) matrices.

**Proof:** Suppose that \((\omega_1(t_0,t_f)) \otimes \omega_2(t_0,t_f)\) is non-singular. Then it is claimed that the kronecker product matrix system \((4.1)\) is completely controllable. Any solution of \((4.1)\) is given by
\[
(y \otimes z)(t) = \int \Psi(t,s) \Phi(t,s) D(t,s)[U(t,s) \otimes U(s,t)]ds + \int \Phi(t,s) D(t,s)[U(t,s) \otimes U(s,t)]ds.
\]
For a given \((n \times 1)\) vector \(y\) and \((m \times 1)\) vector \(z\), choose
\[
(U_1 \otimes U_2)(t) = (D_1 \otimes D_2)(t) \int \Phi(t,s) D(t,s)[U(t,s) \otimes U(s,t)]ds.
\]
Clearly the input signal \((U_1 \otimes U_2)(t)\) is continuous on \(J\) and corresponding general solution of \((4.2)\) with the initial condition 
\[(y \otimes z)(t_0) = (y_0 \otimes z_0)\] is given by
\[
(y \otimes z)(t) = \int \Phi(t,s) D(t,s)[U(t,s) \otimes U(s,t)]ds.
\]
Substituting for \((U_1 \otimes U_2)(t)\) in the definition of \((\omega_1(t_0,t_f)) \otimes \omega_2(t_0,t_f)\), we get
\[
(y \otimes z)(t) = \int \Phi(t,s) D(t,s)[U(t,s) \otimes U(s,t)]ds.
\]
Thus the dynamic system is controllable. This is true for all \(t \in [0, t_f]\), it follows that the kronecker product system is completely controllable.

Next, suppose that the kronecker product system \((4.2)\) is completely controllable on \(J\) and suppose that \((\omega_1 \otimes \omega_2)(t_0,t_f)\) is singular. Then at least one of \(\omega_1\) and \(\omega_2\) must be singular. Suppose \(\omega_1\) is singular then there exist a non-zero \((nx1)\) vector \(y\) such that
\[(y \otimes z)(t_0,t_f) = \int_{t_0}^{t_f} y^* \Phi(t_0,s) D_1(s) U_1(s)ds.
\]
Because of the fact that the integrand in this expression is non-negative continuous function, we have
\[
\left\|y^* \Phi(t_0,s) D(s)\right\| = 0, \text{ se } J.
\]
Thus \(y^* \Phi(t_0,s) D_1(s) = 0, \text{ se } J\).

Since the state equation is completely controllable on \(J\), choose \(y_0 = y\), then there exist a continuous input \(U_1(t)\) such that
\[
0 = \Phi(t_0,t') y + \int_{t_0}^{t_f} \Phi(t',s) D_1(s) U_1(s) ds
\]

Thus, and since \((4.5)\) holds, it follows that \(y^* y = 0\), and this contradicts the fact that \(y \neq 0\). Thus \(\omega_1(t_0,t_f)\) is non-singular. A similar reasoning given above demonstrates that \(\omega_2(t_0,t_f)\) is non-singular and the proof of the theorem is complete.

**References**


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