In this research, we intend to introduce a special operator to derive different symmetric properties of special numbers and two-orthogonal polynomials (Tribonacci Lucas Polynomials). We define a new operator and give some new class of ordinary generating functions of binary bi-orthogonal polynomials and special numbers for boundary value problems.

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**ABSTRACT**

Recently Kasi Viswanadh Kanuri et al. [3] obtained existence and uniqueness of solutions to first order matrix difference system satisfying boundary conditions at three-point. This paper presents a criteria for the existence and uniqueness criteria for fist order fuzzy difference system satisfying initial condition at the initial point.

To establish our main results we make use of the results established on first order difference system by Murty, et.al from [1]. The metrics that we use in this paper are taken from [4]. The results established on initial value problems are taken from [2]. Further results on Fuzzy sets and systems are taken from [5, 6, 7]. An innovative method on the existence of Ψ-bounded solutions for fuzzy dynamical systems on timescales was established by Kasiviswanadh V. Kanuri in [11].

The second order Fibonacci sequence has been generalized in several ways. Some authors have preserved the recurrence relation and altered the first two terms of the sequence while others have preserved the first two terms of the sequence and altered the recurrence relation slightly.

The k-Fibonacci sequence introduced by Falcon [8] depends only on one integer parameter k and is defined as follows:

\[
\begin{align*}
F_n &= kF_{n-1} + F_{n-2}, \text{where } n \geq 2, \quad k \geq 1 \\
F_0 &= 1, F_1 = k
\end{align*}
\]  

(1.3)
The first few terms of this sequence are \( \{1, k, k^2 + 1, k^3 + 2k, \ldots \} \). The particular cases of the \( k \)-Fibonacci sequence are as follows.

If \( k = 1 \), the classical Fibonacci sequence is obtained.

Motivated by the study of \( k \)-Fibonacci numbers in [9], the \( k \)-Lucas numbers have been defined in a similar fashion as

\[
\begin{align*}
L_n = kL_{n-1} + L_{n-2}, \quad & \text{where } n \geq 2, \ k \geq 1, \\
L_0 = 2, L_1 = k
\end{align*}
\]

The first few terms of this sequence are \( \{2, k, k^2 + 2, k^3 + 3k, \ldots \} \).

If \( k = 1 \), the classical Lucas sequence is obtained.

In this part, we define Mersenne numbers, \( k \)-Pell numbers and \( k \)-Jacobsthal-Lucas numbers [Murthy et al.].

For \( n \in \mathbb{N} \), Mersenne numbers, say \( \{M_n\}_{n \in \mathbb{N}} \) is defined recursively by:

\[
\begin{align*}
M_n = 3M_{n-1} - 2M_{n-2}, & \quad \text{for all } n \geq 1, \\
M_0 = 0, M_1 = 1
\end{align*}
\]

For \( n \in \mathbb{N} \), the \( k \)-Pell numbers, denoted by \( \{P_{k,n}\}_{n \in \mathbb{N}} \) defined recursively by

\[
P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \quad n \geq 2, P_{k,0} = 0, P_{k,1} = k \ .
\]

We define \( k \)-Jacobsthal numbers \( \{J_{k,n}\}_{n \in \mathbb{N}} \) by the following recurrence relation as

\[
J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2}, \quad n \geq 2, J_{k,0} = 0, J_{k,1} = 1.
\]

Let \( \alpha(x), \omega_1(x) \) and \( \omega_2(x) \) be the roots of the characteristic equation \( \lambda^3 - x^2 \lambda^2 - x\lambda - 1 = 0 \). Then, the Binet formulas for the Tribonacci and Tribonacci-Lucas polynomials are given by

\[
T_n(x) = -\frac{\alpha^{n+1}(x)}{(\alpha(x) - \omega_1(x))(\alpha(x) - \omega_2(x))} \ , \ n \geq 0,
\]

and

\[
K_n(x) = \alpha^n(x) + \omega_1^n(x) + \omega_2^n(x), \quad n \geq 0,
\]

respectively with \( \alpha(x) = \frac{x^2}{3} + A(x) + B(x), \ \omega_1(x) = \frac{x^2}{3} + \zeta A(x) + \zeta^2 B(x) \) and \( \omega_2(x) = \frac{x^2}{3} + \zeta^2 A(x) + \zeta B(x) \), where

\[
A(x) = \frac{3x^6 + x^3 + 1}{27} + \frac{1}{6} + \frac{1}{2}, \quad B(x) = -\frac{3x^6 + 7x^3 + 1}{27} + \frac{7}{54} + \frac{4}{9}.
\]

On the other hand, the Tribonacci and Tribonacci-Lucas polynomials are defined in a second way by

\[
T_n(x) = S_n(E),
\]

\[
K_n(x) = 3S_n(E) - 2x^2 S_{n-1}(E) - xS_{n-2}(E),
\]

respectively.

Where \( E \) characteristic equation solutions group \( T_n(x) \) is the \( n \)-th Tribonacci polynomial and \( K_n(x) \) is the \( n \)-th Tribonacci Lucas polynomial. In this paper, we will recover the new generating functions of some products of Tribonacci and Tribonacci Lucas polynomials and special numbers.

The further contents of this paper are as follows: Section 1 gives introduction, in section 2, we introduce symmetric functions and some of its properties. We also give some more useful definitions which are used in the following sections. In Section 3 we prove our main result which relates the generating functions of binary products of Tribonacci polynomials and special numbers. Generating functions of binary products of Tribonacci Lucas polynomials and well-known numbers in Section 4.

**Definitions and some Properties**

In this section, we introduce a new symmetric function and give some properties of this symmetric function. We also give some more useful definitions from the literature which are used in the subsequent sections.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet \( A \) is a function of the letters which is invariant under permutation of the letters of \( A \).

\[
\begin{align*}
& \lambda_2(A) = \Pi_{a \in A}(1 + za), \sigma_2(A) = \frac{1}{\Pi_{a \in A}(1 - za)}, \\
& \text{the expansion of which gives the elementary symmetric functions } \Lambda_n(A), \text{ and the complete functions } S_n(A): \\
& \lambda_2(A) = \sum_{n=0}^{+\infty} \Lambda_n(A)z^n, \quad \sigma_2(A) = \sum_{n=0}^{+\infty} S_n(A)z^n.
\end{align*}
\]

Let us now start at the following definition.

Let \( A \) and \( B \) be any two alphabets, then we give \( S_n(A - B) \) by the following form:

\[
\Pi_{a \in A}(1 - zb) = \sum_{n=0}^{+\infty} S_n(A - B)z^n = \sigma_2(A - B).
\]

With the condition \( S_n(A - B) = 0 \) for \( n < 0 \).

Taking \( A = \{0\} \) in (2.1) gives

\[
\Pi_{a \in A}(1 - zb) = \sum_{n=0}^{+\infty} S_n(A - B)z^n = \lambda_2(A - B).
\]

Further, in the case \( A = \{0\} \) or \( B = \{0\} \), we have

\[
\sum_{n=0}^{+\infty} S_n(A - B)z^n = \sigma_2(A) \times \lambda_2(-B).
\]

Thus,

\[
S_n(A - B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B)(\text{see Al}).
\]

Given an alphabet \( A = \{a_1, a_2\} \), the symmetrizing operator \( \delta_{\text{sym}}^{a_1, a_2} \) is defined by

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Given two alphabets \( E \) and \( \mathcal{A} \), we have
\[
\sum_{n=0}^\infty S_n(E)S_n(\mathcal{A})z^n = \frac{1}{1-\sum_{n=0}^\infty S_n(E)S_n(\mathcal{A})z^n}.
\]

Generating Functions of Binary Products of TLP.

Let \( A = \{a_1, a_2\} \) and \( E = \{e_1, e_2, e_3\} \) be two alphabets, we have
\[
\sum_{n=0}^\infty S_n(E)S_{n-1}(a_1 + [-a_2])z^n = \frac{1}{1-\sum_{n=0}^\infty S_n(E)S_{n-1}(a_1 + [-a_2])z^n}.
\]

From (3.4) we deduce
\[
\sum_{n=0}^\infty S_{n-1}(E)S_{n-1}(A)z^n = \frac{1}{1-\sum_{n=0}^\infty S_{n-1}(E)S_{n-1}(A)z^n}.
\]

Given two alphabets \( E \) and \( A \), we have
\[
\sum_{n=0}^\infty S_n(E)S_n(\mathcal{A})z^n = \frac{1}{1-\sum_{n=0}^\infty S_n(E)S_n(\mathcal{A})z^n}.
\]

From (3.4) we deduce
\[
\sum_{n=0}^\infty S_{n-1}(E)S_{n-1}(A)z^n = \frac{1}{1-\sum_{n=0}^\infty S_{n-1}(E)S_{n-1}(A)z^n}.
\]

Given two alphabets \( E \) and \( A \), we have
\[
\sum_{n=0}^\infty S_{n-1}(E)S_{n-1}(A)z^n = \frac{1}{1-\sum_{n=0}^\infty S_{n-1}(E)S_{n-1}(A)z^n}.
\]

Depending on the relationships (3.1) and (3.2), we get the following new results.

For \( n \in \mathbb{N} \), the new generating function of the product of Tribonacci polynomials and k-Fibonacci numbers is given by
\[
\sum_{n=0}^\infty T_n(x)F_{k,n}z^n = \frac{1}{1-zx^2-kx^3}.
\]

Proof. We have \( F_{k,n} = S_n(a_1 + [-a_2]) \). Then, we can see that
\[
\sum_{n=0}^\infty T_n(x)F_{k,n}z^n = \frac{1}{1-zx^2-kx^3}.
\]

For \( n \in \mathbb{N} \), the new generating function of the product of Tribonacci polynomials and k-Fibonacci numbers is given by
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\[
\sum_{n=0}^\infty T_n(x)F_{k,n}z^n = \frac{1}{1-zx^2-kx^3}.
\]
Based on the last propositions (Section 3), we can state the
result

\[ \sum_{n=0}^{\infty} T_n(x)F_{k,n}z^n = \frac{a_2^2 z}{1 - a_2 z} (1 + S_{n-1}(-E) a_2 z + S_n(-E) a_2 z^2) \]

Then, by reduce to same denominator, we obtain the following result

\[ \sum_{n=0}^{\infty} K_n(x)F_{k,n}z^n = 3 \sum_{n=0}^{\infty} T_n(x)F_{k,n}z^n - \frac{p_1(x)z + p_2(x)z^2 + p_3(x)z^3 + q_1(x)z^4 + q_2(x)z^5 + q_3(x)z^6}{1 + q_1(x)z + q_2(x)z^2 + q_3(x)z^3 + q_4(x)z^4 + q_5(x)z^5 + q_6(x)z^6} \]

For \( n \in \mathbb{N} \), the new generating function of the product of

Trionacci polynomials and k-Pell numbers is given by

\[ \sum_{n=0}^{\infty} K_n(x)F_{k,n}z^n = \frac{a_2^2 z}{1 - a_2 z} (1 + S_{n-1}(-E) a_2 z + S_n(-E) a_2 z^2) \]

After a simple calculation, of \( p_1(x) \) and \( q_1(x) \) we obtain (3.8).

Based on the last propositions (Section 3), we can state the

new following theorems.

For \( n \in \mathbb{N} \), the new generating function of the product of

Trionacci Lucas polynomials and k-Fibonacci numbers is given by

\[ \sum_{n=0}^{\infty} K_n(x)F_{k,n}z^n = \frac{a_2^2 z}{1 - a_2 z} (1 + S_{n-1}(-E) a_2 z + S_n(-E) a_2 z^2) \]

This completes the proof.
\[
\sum_{n=0}^{\infty} K_n(x)F_{k,n} z^n = \frac{3k}{(a_1 + a_2)} \left( \sum_{n=0}^{\infty} S_n(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_{n-1}(E)(a_2 z)^n \right) + \frac{2kz^2}{(a_1 + a_2)} \left( \sum_{n=0}^{\infty} S_{n-1}(E)(a_2 z)^n - \sum_{n=0}^{\infty} S_{n-2}(E)(a_1 z)^n \right) + \frac{kz}{a_1 z} \left( \sum_{n=0}^{\infty} S_n(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_{n-1}(E)(a_2 z)^n \right)
\]

\[
2 \sum_{n=0}^{\infty} K_n(x) F_{k,n} z^n + \frac{1}{(a_1 + a_2)} \frac{1}{1 + \frac{z}{S_1(-E)} \frac{a_1 z}{S_2(-E) a_2 z^2 - S_3(-E) a_1 z^2}}
\]

\[
\sum_{n=0}^{\infty} K_n(x) L_{k,n} z^n = 2 \sum_{n=0}^{\infty} K_n(x) F_{k,n} z^n - \frac{1}{k} \left( \frac{1}{1 + S_1(-E) a_1 z + S_2(-E) a_1 z^2 + S_3(-E) a_1 z^3} \right)
\]

where

\[
p_1(x) = -3S_1(-E) - 2x^2,
p_2(x) = -3(\alpha_1 - \alpha_2) S_2(-E) + x(a_1 - a_2),
p_3(x) = -3(\alpha_1 - \alpha_2)^2 + 2x^2 a_1 a_2 S_2(-E) - x a_1 a_2 S_1(-E),
p_4(x) = -2x^2(\alpha_1 - \alpha_2) a_1 a_2 S_3(-E),
p_5(x) = x a_1^2 a_2^2 S_3(-E),
\]

and

\[
q_1(x) = (a_1 - a_2) S_1(-E),
q_2(x) = S_2(-E)(a_1 - a_2)^2 - a_1 a_2 S_1(-E) - 2S_2(-E),
q_3(x) = S_2(-E)(a_1 - a_2)^3 - a_1 a_2 \left( a_1 - a_2 \right) S_3(-E),
q_4(x) = -a_1 a_2 (a_1 - a_2)^2 S_3(-E) + a_1^2 a_2^2 (S_2(-E) - 2S_3(-E) S_1(-E)),
q_5(x) = a_1^2 a_2^2 S_3(-E) S_1(-E),
q_6(x) = -S_1(-E) a_1 a_2 S_3(-E)
\]

After a simple calculation, of \( p_1(x) \) and \( q_1(x) \) we obtain (4.2).

For \( n \in \mathbb{N} \), the new generating function of the product of Tribonacci Lucas polynomials and k-Pell numbers is given by the following sum:

\[
\sum_{n=0}^{\infty} K_n(x) P_{k,n} z^n = \frac{x^2 z + 4x z^2 + (12 + 3k + k^2)x z^3 + 4k x z^4 + k^2 x z^5}{1 - 2x^2 z + (2x + k^2) z^2 - 2(3k + 4k x + k^2) z^3 - k(k + 1)x z^4 + 2k^2 x z^5 - k^3 x z^6}
\]

This completes the proof.

References


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