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Research Article

A STRUCTURED APPROACH TO A PERTURBATION OF HAMILTONIAN SYSTEM WITH PERIODIC COEFFICIENTS

Traoré G. Y. AROUNA* and Mouhamadou DOSSO

UFR Mathématiques et Informatiques, Université Félix Houphouët Boigny of Cocody-Abidjan, Côte d' Ivoire

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ABSTRACT

This paper deals with a structured approach to a perturbation of Hamiltonian systems with periodic coefficients. From a perturbation theory proposed by T.G.Y. Arouna, *et al*., a certain type of perturbation of Hamiltonian systems with periodic coefficients is presented. Two numerical examples are given to confirm our theoretical results.

INTRODUCTION

Let $J^* = -J$ be an invertible $2N \times 2N$ matrix. In this paper, we study a perturbation of Hamiltonian systems with periodic coefficients. These types of systems generally model the problems of engineering and physics (Yakubovich and Starzhinskii [15]). Particularly, they come from the theory of optimal control (Brezinski [3]), and parametric resonance (Yakubovich and Starzhinskii [15]). These systems are linear differential systems of the form (1)

$$J\frac{dX(t)}{dt} = H(t)X(t), \tag{1}$$

where $H(t) = H(t + P) = H(t) \in \mathbb{R}^{2N\times 2N}$, $\forall t \in \mathbb{R}$ and $P > 0$.

The fundamental solution $X(t)$ of

$$\begin{cases} J\frac{dX(t)}{dt} = H(t)X(t) \\ X(0) = I_{2N} \end{cases}, \tag{2}$$

is symplectic i.e. $X^T(t)JX(t) = J$. Let A be a symmetric matrix, i.e. $A = A^*$ is Hermitian and $\mathcal{H}$ be a Hamiltonian matrix i.e. $(J\mathcal{H}) = (J\mathcal{H})^*$. The symmetric matrices and the Hamiltonian matrices possess simple and useful properties, among which we recall the following results (Batzke in [2] and Roge in [11]) useful for the following. Here and in what follows, the symbol $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$.

Corresponding author:* **Traoré G. Y. AROUNA
UFR Mathématiques et Informatiques, Université Félix Houphouët Boigny of Cocody-Abidjan, Côte d' Ivoire

Corollary 1 *(Takagi's factorization)[11] If $A \in M_N(\mathbb{F})$ is symmetric then there exists a unitary $U \in M_N(\mathbb{F})$ and a real nonegative diagonal matrix $\Sigma = diag(\sigma_1, \cdots, \sigma_N)$ such that $A = U\Sigma U^T$.*

Using corollary1 and the fact that for any Hamiltonian matrix $\mathcal{H}$, the matrix $\mathcal{H}J^{-1}$ is symmetric, we have the following result

Lemma 1 *[2] Let $\mathcal{H} \in \mathbb{F}^{2N\times 2N}$ be a J-Hamiltonian matrix of rank k, and $J \in \mathbb{F}^{2N\times 2N}$* be *anti-symmetric and invertible matrix.*

1. If $\mathbb{F} = \mathbb{R}$, then there exists a matrix $U \in \mathbb{R}^{2N\times k}$ of rank k and a diagonal matrix $\Sigma = diag(s_1, \dots, s_k)$, *with* $s_j \in \{\pm 1\}, \forall j = 1, \dots, k$ such that $\mathcal{H} = U\Sigma U^T J$.
2. If $\mathbb{F} = \mathbb{C}$, then there exists a matrix $U \in \mathbb{C}^{2N\times k}$ of rank k *such that* $\mathcal{H} = UU^*J$.

The Hamiltonian matrices often lead us to special case of vector space:''isotrpic subspaces'' ([1],[5], [7], [8], [10], [12], [13]) which are defined as following.

Definition 1

A subspace $X \subseteq \mathbb{R}^{2N}$ is called isotropic if $X \perp JX$. The maximum isotropic subspace is called Lagrangian subspace.
Clearly a subspace $\mathcal{L}$ is isotropic if and only if any matrix L, whose columns generate $\mathcal{L}$ satisfies $rank(L) \le N$ and $L^*JL = 0$, in particular, if in addition to the condition $L^*JL = 0$, we have $rank(L) = N$, in this case L is said to be J-Lagrangian (see for example in [1], [5], [7], [8], [10], [12], [13]).

The purpose of this present article is to link any unperturbed symplectic matrix W with its rank-k (with $k \le N$) perturbation $\widetilde{W} = W + E$ $(E \in \mathbb{F}^{2N\times 2N})$ that preserves its J-symplecticity structure and to determine the types of perturbations of (2). whose solutions $\widetilde{X}(t)$ are the rank-k perturbations of $X(t)$ matrix solution of (2). Thus to understand our study, we give a generalization of lemma 7.1 of Mehl et al [9] and the conditions for which any rank-$k(k \le N)$ perturbation of symplectic matrices remain symplectic in section 2. In section 3, we present the case of a certain type of perturbation of Hamiltonian systems with periodic coefficients. Finally, before approaching the last section, where we give a concluding remarks, we give two numerical examples in section 4 in order to check our theoretical results. Throughout the paper, instead of the term "J-symplectic", we will sometimes say "symplectic". The symbol $\|.\|$ denotes the Euclidean norm of matrices or vectors. The identity (respectively null) matrix of order is denoted by I_{2N} (respectively 0_{2N}) or just I (respectively 0) if the order is clear from the context. Finally, the transpose of a matrix (or vector) U is denoted by U^T if $\mathbb{F} = \mathbb{R}$ or $U^\star$ if $\mathbb{F} = \mathbb{C}$.

A simplified form of a perturbation of symplectic matrices preserving its symplectic structure

Let $W \in \mathbb{F}^{2N\times 2N}$ be a symplectic matrix. To proof proposition 2, let's start with the following result

Proposition 1 Let $\widetilde{W} = W + E$, where $E \in \mathbb{F}^{2N\times 2N}$ is such that $E^*JE = 0$. Then $\widetilde{W}$ is J-symplectic if and only if EW^{-1} is J-Hamiltonian.

Proof
Suppose that $\widetilde{W}$ is J-symplectic. We know that $\widetilde{W}$ can be in the form $\widetilde{W}$ = (I+EW^{-1})W, so I+EW^{-1} is also J-symplectic. In fact

$$\begin{aligned}
&(I + EW^{-1})^\star J (I + EW^{-1}) = J \\
&\Leftrightarrow (J + W^{-\star}E^\star J)(I + EW^{-1}) = J \\
&\Leftrightarrow J + W^{-\star}E^\star J + JEW^{-1} + W^{-\star}\underbrace{E^\star JE}_{=0}W^{-1} = J \\
&\Rightarrow W^{-\star}E^\star J + JEW^{-1} = 0 \\
&\Leftrightarrow (JEW^{-1})^\star - JEW^{-1} = 0,
\end{aligned}$$

So EW^{-1} is J-Hamiltonian.

Conversely, suppose EW^{-1} is J-Hamiltonian. Since

$$\begin{aligned}(I+EW^{-1})^{\star}\,J\,(I+EW^{-1}) &= (J+W^{-\star}E^{\star}J)\,(I+EW^{-1})\\ &= J+W^{-\star}E^{\star}J+JEW^{-1}+W^{-\star}\underbrace{E^{\star}JE}_{=0}W^{-1}\\ &= J\underbrace{-(JEW^{-1})^{\star}+JEW^{-1}}_{=0}=J,\ \text{because } EW^{-1} \text{ is } J\text{-Hamiltonian.}\end{aligned}$$

So $I+EW^{-1}$ is symplectic. Therefore $\widetilde{W} = (I+EW^{-1})W$ is symplectic as a product of symplectic matrices.

Now, we can give a generalization of lemma 7.1 of Mehl *et al* [9] in the proposition below

Proposition 2 If $\widetilde{W} = W + E(E \in \mathbb{F}^{2N\times 2N})$ is *symplectic* such that $E^{*}JE = 0$ *and rank of* E *is* k *(with* $\leq N$*), then there is a matrix* $R \in \mathbb{F}^{2N\times k}$ *of rank k, whose columns generate an isotropic subspace such that*

$$\widetilde{W} = \begin{cases}(I+RR^{*}J)W, & si\ \mathbb{F}=\mathbb{C}\\ (I+R\Sigma R^{T}J)W, & si\ \mathbb{F}=\mathbb{R}'\end{cases} \tag{3}$$

where $\Sigma = diag\,(s1, \ldots, sk)$, *with* $s_j \in \{\pm 1\}, \forall\, j = 1, \ldots, k$.
Conversely, for any matrix $R \in \mathbb{F}^{2N\times 2N}$ of rank k, whose columns generate an isotropic subspace, the matrix $\widetilde{W}$ is symplectic.

Proof
Since $\widetilde{W}$ is symplectic such that $rank(E) = k \leq N$ and $E^{*}JE = 0$, then according to proposition1, the matrix $EW{-}1$is J-Hamiltonian.

If $\mathbb{F} = \mathbb{C}$, we have $EW^{-1} \in \mathbb{C}^{2N\times 2N}$. Then, according to lemma 1, there is a matrix $R \in \mathbb{C}^{2N\times k}$ of rank k verifying the following condition

$$EW^{-1} = RR^{*}J, \text{(Batzke in [2])} \tag{4}$$

this implies that $E = RR^{*}JW$. Hence $\widetilde{W}$ can be written down as $\widetilde{W} = (I + RR^{*}J)W$. Now let's show that the columns of R belong to an isotropic subspace.

Since $I + RR^{*}J$ is symplectic, we have :

$$\begin{aligned}(I+RR^{\star}J)^{\star}\,J\,(I+RR^{\star}J) &= J\\ \Leftrightarrow (J-JRR^{\star}J)\,(I+RR^{\star}J) &= J\\ J\underbrace{-JRR^{\star}J+JRR^{\star}J}_{=0}-JRR^{\star}JRR^{\star}J &= J\\ \Rightarrow -JRR^{\star}JRR^{\star}J &= 0\\ \Rightarrow RR^{\star}JRR^{\star} &= 0, \qquad (5)\end{aligned}$$

Multiplying the two sides of equation (5) respectively on the left by R^{*} and on the right by R, we get $R^{*}R[R^{*}JR]R^{*}R = 0$, which implies $R^{*}JR = 0$, because $R^{*}R$ is invertible. Let's put $R = [r_1, \ldots, r_k]$, then :

$$\begin{aligned}R^{\star}JR &= 0\\ \Leftrightarrow \begin{bmatrix} r_1^{\star}J\\ \vdots\\ r_k^{\star}J\end{bmatrix}[r_1,\ldots,r_k] &= 0\\ \Leftrightarrow \begin{bmatrix} r_1^{\star}Jr_1 & r_1^{\star}Jr_2 & \cdots & r_1^{\star}Jr_k\\ \vdots & \vdots & \ddots & \vdots\\ r_k^{\star}Jr_1 & r_k^{\star}Jr_2 & \cdots & r_k^{\star}Jr_k\end{bmatrix} &= 0_{k\times k}\end{aligned}$$

This shows that $r_l^{*}Jr_j = 0, \forall\, j, l \in \{1, \ldots, k\}$. Hence taking $\mathcal{N} = span[r, \ldots, r_k]$, it follows that $\mathcal{N} \perp J\mathcal{N}$. Which means that the columns of R generate an isotropic subspace.

If $\mathbb{F} = \mathbb{R}$, then in this case $EW^{-1} \in \mathbb{R}^{2N\times 2N}$, so according to lemma 1, *t*here exists a matrix $R \in \mathbb{R}^{2N\times k}$ of rank k and a diagonal matrix $\Sigma = diag\,(s1, \ldots, sk)$, *with* $s_j \in \{\pm 1\}, \forall\, j = 1, \ldots, k$ *such that:*

$$EW^{-1} = R\Sigma R^{T}J \text{ (Batzke in [2])} \tag{6}$$

this implies that $E = R\Sigma R^{T}JW$. *Replacing E by its expression in that of* $\widetilde{W}$, *we obtain* $\widetilde{W} = (I + R\Sigma R^{T})W$. *Reasoning in a similar way such as the complex case, we deduct that the columns of matrix* R *generate an isotropic subspace.*

Conversely, it's easy to see that for any matrix $R \in \mathbb{F}^{2N\times k}$ of rank k, whose columns generate an isotropic subspace, the matrix $\widetilde{W}$ is symplectic. Indeed, it suffices to note that $I + RR^{*}J$ is J-symplectic, which is immediate from

$$(I + RR^{\star}J)^{\star} J (I + RR^{\star}J) = J - JRR^{\star}J + JRR^{\star}J - JR\underbrace{R^{\star}JR}_{=0}R^{\star}J = J.$$

For a symplectic matrix W and a matrix $U \in \mathbb{F}^{2N\times k}$ of rank k, whose columns generate an isotropic subspace, considering the set $\mathcal{E}(W, U)$ of rank-$\tilde{k}$ (with $\tilde{k} \leq k$) perturbations of symplectic matrices defined by

$\mathcal{E}(W, U) = \{\widetilde{W}\ symplectic : \widetilde{W} = W + UA^{*}, \text{with } A \in \mathbb{F}^{2N\times k}\}$, we have the following remark

Remark 1

- For all $\widetilde{W} \in \mathcal{E}(W, U)$, there exists a matrix $R \in \mathbb{F}^{2N\times k}$ *of rank* $\tilde{k}$, *whose columns generate an isotropic subspace such that*

$$\widetilde{W} = \begin{cases} (I + RR^{*}J)W, & si\ \mathbb{F} = \mathbb{C} \\ (I + R\Sigma R^{T}J)W, & si\ \mathbb{F} = \mathbb{R}' \end{cases} \tag{7}$$

where $\Sigma = diag(s_1,...,s_{\tilde{k}})$, *with* $s_j \in \{\pm 1\}, \forall\, j = 1, \dots, \tilde{k}$. Because for all $\widetilde{W} \in \mathcal{E}(W, U)$, $\widetilde{W}$ is symplectic and there exists $A \in \mathbb{F}^{2N\times k}$, such that $\widetilde{W} = W + UA^{*}$, with $U^{*}JU = 0$ and $\text{rank}(UA^{*}) = \tilde{k}$.

- $\mathcal{E}(W, U)$ is a non-empty set, because $W \in \mathcal{E}(W, U)$.

Consider the *J*-symplectic matrix function $(X(t))_{t\in\mathbb{R}}$. In particular, if $X(t)$ is the solution of (2), then according to proposition 2, any element of $\mathcal{E}\,(X(t), U)$ can be in the form

$$\tilde{X}(t) = \begin{cases} (I + R(t)R^{*}(t)J)X(t), & si\ \mathbb{F} = \mathbb{C} \\ (I + R(t)\Sigma(t)R^{T}(t)J)X(t), & si\ \mathbb{F} = \mathbb{R}' \end{cases} \tag{8}$$

Starting from this concept, we can determine a link between the rank-*k* (with $k \leq N$) perturbation of Hamiltonian systems and a certain perturbation of differential system.

Case of certain types of perturbation of Hamiltonian systems with periodic coefficients

Let's $U \in \mathbb{R}^{2N\times k}$ (with $k \leq N$) be a constant matrix of rank *k* such that its columns belong to an isotropic subspace. Consider the perturbed Hamiltonian equation

$$J\frac{d\tilde{X}(t)}{dt} = [H(t) + E(t)]\tilde{X}(t), \tag{9}$$

where $H(t + P) = H^{T}(t) = H(t) \in \mathbb{R}^{2N\times 2N}$, $E(t + P) = E^{T}(t) = E(t) \in \mathbb{R}^{2N\times 2N}, \forall\, t \in \mathbb{R}$ and $P > 0$.

We have the following proposition

Proposition 3 *If a solution* $\tilde{X}(t)$ *of (9) is symplectic such that rank(*$\tilde{X}(t) - X(t)) = k \leq N$ *and* $(\tilde{X}(t) - X(t))^{T}J\left(\tilde{X}(t) - X(t)\right) = 0$, $\forall t \in \mathbb{R}$. *Then there exists a constant matrix* $R \in \mathbb{R}^{2N\times k}$ *of rank k, whose columns generate an isotropic subspace and a diagonal matrix* $\Sigma = diag\,(s_1, \dots, s_k)$, *with* $sj = \pm 1$ *for all* $\boldsymbol{j} = 1,\dots, k$ *so that* $E(t)$ *be of the form*

$$E(t) = \left(JR\Sigma R^{T}H(t)\right)^{T} + JR\Sigma R^{T}H(t) + (R\Sigma R^{T}J)^{T}H(t)(R\Sigma R^{T}J),\ \ \forall\, t \in \mathbb{R} \tag{10}$$

Proof Let $t \in \mathbb{R}$. Suppose $\tilde{X}(t)$ is symplectic such that *$rank(\tilde{X}(t) - X(t)) = k \leq N$ and*
$(\tilde{X}(t) - X(t))^T J\left(\tilde{X}(t) - X(t)\right) = 0$, $\forall t \in \mathbb{R}$. Then according to proposition 2, there is a matrix function $R(t) \in \mathbb{R}^{2N\times k}$of rank k, whose columns belong to an isotropic subspace and a diagonal matrix $\Sigma(t)=diag(s_1(t),\ldots,s_k(t))$, *with* $s_j(t) = \pm 1$ *for all* $j = 1,\ldots,k$ *such that* $\tilde{X}(t) = (I + R(t)\Sigma(t)R^T(t)J)X(t)$ *because*
$\tilde{X}(t) \in \mathbb{R}^{2N\times 2N}$. *Now let's show that* $R(t)$ and $\Sigma(t)$ are constant matrices.

$$\begin{aligned}
\tilde{H}(t)\tilde{X}(t) &= J\frac{d\tilde{X}(t)}{dt}\\
&= J\left(I + R(t)\Sigma(t)R^T(t)J\right)\frac{dX(t)}{dt} + J\left[\frac{d\left(R(t)\Sigma(t)R^T(t)\right)}{dt}\right]JX(t)\\
&= J\left(I + R(t)\Sigma(t)R^T(t)J\right)J^{-1}H(t)X(t) + J\left[\frac{d\left(R(t)\Sigma(t)R^T(t)\right)}{dt}\right]JX(t)\\
&= \left[\left(I + JR(t)\Sigma(t)R^T(t)\right)H(t) + J\frac{d\left(R(t)\Sigma(t)R^T(t)\right)}{dt}J\right]\left(I - R(t)\Sigma(t)R^T(t)J\right)\tilde{X}(t)\\
&= \left[\underbrace{\left(I - R(t)\Sigma(t)R^T(t)J\right)^T H(t)\left(I - R(t)\Sigma(t)R^T(t)J\right)}_{=\tilde{H}(t)}\right]\tilde{X}(t)\\
&+ \left[J\frac{d\left(R(t)\Sigma(t)R^T(t)\right)}{dt}J\left(I - R(t)\Sigma(t)R^T(t)J\right)\right]\tilde{X}(t)
\end{aligned}$$

This implies that $J\frac{d\left(R(t)\Sigma(t)R^T(t)\right)}{dt}J(I - R(t)\Sigma(t)R^T(t)J) = 0$, since $\tilde{X}(t)$ is a solution of (9), which is a Hamiltonian system with periodic coefficients. Hence $\frac{d\left(R(t)\Sigma(t)R^T(t)\right)}{dt} = 0$, because J and $(I - R(t)\Sigma(t)R^T(t)J)$ are invertible. Thus $\exists\, C \in \mathbb{R}$ such that $R(t)\Sigma(t)R^T(t) = C$. Putting $C = R(0)\Sigma(0)R^T(0)$, $\tilde{X}(t)$ becomes $\tilde{X}(t) = (I + R\Sigma R^T J)X(t)$, with $R = R(0)$, and $\Sigma = \Sigma(0)$. Replacing the new expression of $\tilde{X}(t)$ in the first member of (9), we obtain:

$$\begin{aligned}
J\frac{d\tilde{X}(t)}{dt} &= J(I + R\Sigma R^T J)J^{-1}J\frac{dX(t)}{dt}\\
&= J(I + R\Sigma R^T J)J^{-1}H(t)X(t), \quad \text{according form system} \quad (2)\\
&= [H(t) + JR\Sigma R^T H(t)](I + R\Sigma R^T J)^{-1}\tilde{X}(t)\\
&= \left[H(t) + \underbrace{(JR\Sigma R^T H(t))^T + JR\Sigma R^T H(t) + (R\Sigma R^T J)^T H(t)(R\Sigma R^T J)}_{E(t)}\right]\tilde{X}(t)
\end{aligned}$$

Hence system (9), where

$$E(t) = \left(JR\Sigma R^T H(t)\right)^T + JR\Sigma R^T H(t) + (R\Sigma R^T J)^T H(t)(R\Sigma R^T J).$$

From this proposition, we deduce the following corollary

Corollary 2 *If there exists a P-periodic and symmetric matrix $E(t)$ such that the fundamental $\tilde{X}(t)$ of the Hamiltonian equation*

$$J\frac{d\tilde{X}(t)}{dt} = [H(t) + E(t)]\tilde{X}(t), \quad \tilde{X}(0) = I + UU^T J \tag{11}$$

belongs to $\mathcal{E}(X(t), U)$, then $E(t)$ is of the form (10).

Proof Since the solution $\tilde{X}(t)$ of (12) belongs to $\mathcal{E}(X(t), U)$, then there is a matrix $A \in \mathbb{R}^{2N\times k}$ such that $\tilde{X}(t) = X(t) + UA^*$ is symplectic and $rank(UA^*) = \tilde{k} \leq k$, with $(UA^*)^* JUA^* = 0$. Hence, according to proposition 3, there exists a matrix $R \in \mathbb{R}^{2N\times k}$ of rank $\tilde{k}$, whose columns belong to an isotropic subspace such that $E(t)$ be of the form (10).

The following corollary is a generalization of proposition 4.1 of Arouna et al in [1].

Corollary 3 *Consider the Hamiltonian equation (9), with*

$$E(t) = \left(J\,U\Sigma U^T H(t)\right)^T + J\,U\Sigma U^T H(t) + \left(U\Sigma U^T J\right)^T H(t)(U\Sigma U^T J), \ \forall\, t \in \mathbb{R},$$

where $U \in \mathbb{R}^{2N\times k}$ is a matrix of rank k, whose columns belong to an isotropic subspace and $\Sigma = \mathrm{diag}(s_1,\ldots,s_k)$, with $s_j \in \{\pm 1\}$, $\forall\, j = 1,\ldots,k$. Then $\tilde{X}(t) = (I + U\Sigma U^T J)X(t)$ is a solution of (9).

Proof By derivation of $\tilde{X}(t)$, we get

$$\begin{aligned} J\frac{d\tilde{X}(t)}{dt} &= J(I+U\Sigma U^TJ)J^{-1}J\frac{dX(t)}{dt} \\ &= J(I+U\Sigma U^TJ)J^{-1}H(t)X(t), \quad \text{according form system (2)} \\ &= [H(t)+JU\Sigma U^TH(t)](I+U\Sigma U^TJ)^{-1}\tilde{X}(t) \\ &= \left[H(t)+\underbrace{(JU\Sigma U^TH(t))^T+JU\Sigma U^TH(t)+(U\Sigma U^TJ)^TH(t)(U\Sigma U^TJ)}_{E(t)}\right]\tilde{X}(t) \end{aligned}$$

Hence system (9), where $E(t) = (JU\Sigma U^T H(t))^T + JU\Sigma U^T H(t) + (U\Sigma U^T J)^T H(t)(U\Sigma U^T J)$. So $\tilde{X}(t)$ is a solution of (9).

Algorithm and Numerical examples

From corollary 1 and the algorithms of Fuller in [6] and Wei et al in [14], we propose the following algorithm which leads us to factorize any rank-k (with $k \leq N$) perturbation of symplectic matrix, as proposed in lemma 1.

Algorithm 1

Input: *A: Symmetric $2N$ by $2N$ matrix of rank k, with $k \leq N$.*
Output: $C \in \mathbb{F}^{2N\times k}$*: nonsigular matrix of rank k and diagonal matrix* $L = \mathrm{diag}(s_1, \ldots, s_k) \in \mathbb{F}^{2N\times k}$, with $s_j \in \{\pm 1\}$, $\forall$ j = 1, …, k *of rank k such that* $A = CLC^T$ *if* $\mathbb{F} = \mathbb{R}$ *and* $A = CC^*$ *if* $\mathbb{F} = \mathbb{C}$.

If $A \in \mathbb{R}^{2N\times 2N}$, *then (see* Fuller in *[6])*

- *Diagonalize matrix A, ie find a non-singular matrix C_1 and a diagon*al matrix D *such that* $C_1^T AC_1 = D$.
- *Store the elements of the diagonal matrix D in descending order to obtain another diagonal matrix* Σ and put $C = C_1(\Sigma^{\frac{1}{2}})^T(:,1{:}k)$ such that $C^TAC = L$, ($L = \mathrm{diag}(s_1, \ldots, s_k) \in \mathbb{F}^{2N\times k}$, with $s_j \in \{\pm 1\}$, $\forall$ j = 1, …, k *and* $rank(L) = k$).

Else

- **Step 1:** *Reduce A to a complex symmetric tridiagonal form, i.e.($P^*AP = M$).*
- **Step 2:** *Applied algorithm* 3.1 *of* Wei et al in *[14] to compute the Takagi's factorization of the complex symmetric tridiagonal matrix M, i.e.($M = Q\Sigma Q^*$).*
- **Step 3:** *Combine the two previous steps to obtain the Takagi factorization of A, i.e,($A = P(Q\Sigma Q^*)P^* = C_1\Sigma C_1^*$ with $C_1 = PQ$).*
- *Put* $C = C_1(\Sigma^{1/2}(:,1:k))^\star$ *to get* $A = CC^*$.

End

Now, we present numerical examples to confirm our theoretical results. Here, all experiments are done with MATLAB7.9.0 (R2009b).

Example 1 *Consider the following symplectic matrix function*

$$W(t) = \begin{bmatrix} I_3 & Y(t) \\ 0_3 & I_3 \end{bmatrix}, \ where\ Y(t) = \begin{bmatrix} sin(t) & cos(t) & 1 \\ cos(t) & 1 & 0 \\ 1 & 0 & 1+sin(t) \end{bmatrix} \ and\ t \in \mathbb{R}.$$

Let's put $\tilde{W}(t) = W(t) + E(t)$, where $E(t) = \begin{bmatrix} 0 & A(t) \\ 0 & 0 \end{bmatrix}$, with $A^T(t) = A(t) \in \mathbb{R}^{2N\times k}(k \leq 3)$. *It is easy to check that* $E^T(t)J\,E(t) = 0$, $\forall\, t \in \mathbb{R}$. *Indeed* $\forall\, t \in \mathbb{R}$, *we have:*

$$E^T JE = \left[\begin{pmatrix} 0 & A(t) \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & A(t) \\ 0 & 0 \end{pmatrix} \right] = \left[\begin{pmatrix} 0 & 0 \\ A^T(t) & 0 \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & A(t) \\ 0 & 0 \end{pmatrix} \right]$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & -A^T(t) \end{pmatrix} \begin{pmatrix} 0 & A(t) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The purpose of this example is to show that for a given symplectic matrix $\widetilde{W} = W + E$, *where* W *is symplectic and* $E \in \mathbb{R}^{2N \times 2N}$ *is such that* $rank(E) = k$ (*with* $k = 2, 3$) *and* $E^T JE = 0$, *then there is a matrix* $U \in \mathbb{R}^{2N \times k}$ *of rank* k, *whose columns belong to an isotropic subspace and a diagonal matrix* $\Sigma = diag\,(s_1, s_2, s_3)$, *with* $s_j = \pm 1$ *for all* $j = 1, 2, 3$; *such that* $\widetilde{W} = (I + U\Sigma U^T J)\, W$.

- *Taking* $A(t) = \begin{pmatrix} 2\cos(2t) & -1 & 0 \\ -1 & 2\cos(2t) & -1 \\ 0 & -1 & 2\cos(2t) \end{pmatrix}$, *then at* $t = 0.293$ *for example, we get:*

$$E(0.293) = \begin{pmatrix} 0 & 0 & 0 & -1.6663 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 & -1.6663 & 1.0000 \\ 0 & 0 & 0 & 0 & 1.0000 & -1.6663 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with $rank\big(E(0.293)\big) = 3$. *The computation of* $\widetilde{W}(0.293)$ *gives*

$$\widetilde{W}(0.293) = \begin{pmatrix} 1.0000 & 0 & 0 & -1.3775 & 1.9574 & 1.0000 \\ 0 & 1.0000 & 0 & 1.9574 & -0.6663 & 1.0000 \\ 0 & 0 & 1.0000 & 1.0000 & 1.0000 & -0.3775 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \end{pmatrix} \quad verifying\ \|\widetilde{W}^T(0.293) J \widetilde{W}(0.293) - J\| = 0$$

and using Algorithm 1 to

$$E(JW)^{-1}(0.293) = \begin{pmatrix} 1.6663 & -1.0000 & 0 & 0 & 0 & 0 \\ -1.0000 & 1.6663 & -1.0000 & 0 & 0 & 0 \\ 0 & -1.0000 & 1.6663 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we get $U = \begin{bmatrix} -0.8776 & -0.9128 & 0.2511 \\ 1.2411 & 0.0000 & 0.3550 \\ -0.8776 & 0.9128 & 0.2511 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

of rank 3 *and* $\Sigma = I_3$ *satisfying the following equality*

$||U^T JU|| = 0$ *and* $||\,\widetilde{W}(0.293) - (I + U\Sigma U^T J) W(0.293)|| = 2.8486 \times 10^{-15}$.

This shows that there exists a matrix $U \in \mathbb{R}^{6\times3}$ of rank 3, whose columns belong to an isotropic subspace and a diagonal matrix $\Sigma = I_3$ such that $\widetilde{W}(0.293) \equiv (I + U\Sigma U^T J) W(0.293)$.

- In this *latter example, we take* $A(t) = \begin{pmatrix} 2\sin(t) + 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Thus at* $t = 1.934$ *for example, we get:*

$$E(1.934) = \begin{pmatrix} 0 & 0 & 0 & -2.8695 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with $rank(E(1.934)) = 2$. *The computation of* $\widetilde{W}(1.934)$ *gives*

$$\widetilde{W}(1.934) = \begin{pmatrix} 1.0000 & 0 & 0 & -1.9348 & -0.3553 & 1.0000 \\ 0 & 1.0000 & 0 & -0.3553 & 0 & 0 \\ 0 & 0 & 1.0000 & 1.0000 & 0 & 1.9348 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \end{pmatrix} \text{ satisfying } \|\widetilde{W}^T(1.934) J \widetilde{W}(1.934) - J\| = 0.$$

The application of Algorithm 1 to

$$E(JW)^{-1}(1.934) = \begin{pmatrix} 2.8695 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

yields matrix

$$U = \begin{bmatrix} 1.6940 & 0 \\ 0 & 1.0000 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

of rank 2 and $\Sigma = I_2$ *verifying the conditions bellow*

$$||U^T J U|| = 0 \text{ and } ||\widetilde{W}(1.934) - (I + U\Sigma U^T J)W(1.934)|| = 4.4409 \times 10^{-16}.$$

This shows that there exists a matrix $U \in \mathbb{R}^{6\times 2}$ of rank 2, whose columns belong to an isotropic subspace and a diagonal matrix $\Sigma = I_2$ such that $\widetilde{W}(1.934) \equiv (I + U\Sigma U^T J)W(1.934)$.

Example 2 *In this example, consider system (9), with*

$$H(t) = \begin{pmatrix} P(t) & 0_3 \\ 0_3 & I_3 \end{pmatrix}$$

and

$$E(t) = \begin{bmatrix} 0 & 0 & 0 & a\cos(\sqrt{7}t) + d & 0 & e\cos(2\sqrt{7}t) \\ 0 & 0 & 0 & 0 & \alpha & e\sin(2\sqrt{7}t) \\ 0 & 0 & 0 & b\cos(2\sqrt{7}t) & g\sin(5\sqrt{7}t) & \eta \\ a\cos(\sqrt{7}t) + d & 0 & b\cos(2\sqrt{7}t) & \beta_1\cos(\sqrt{7}t) + d_1 & 0 & \beta_2\cos(2\sqrt{7}t) \\ 0 & \alpha & g\sin(5\sqrt{7}t) & 0 & \delta_1 & \alpha_1\sin(2\sqrt{7}t) \\ e\cos(2\sqrt{7}t) & e\sin(2\sqrt{7}t) & \eta & \beta_2\cos(2\sqrt{7}t) & \alpha_1\sin(2\sqrt{7}t) & 0 \end{bmatrix},$$

where

$$P(t) = \begin{pmatrix} 4 + \epsilon\cos(\gamma t) & 0 & \delta\cos(2\gamma t) \\ 0 & 3 & \epsilon\sin(5\gamma t) \\ \delta\cos(2\gamma t) & \epsilon\sin(5\gamma t) & 2 \end{pmatrix}, \text{ with } \gamma = \sqrt{7}$$

and $a, b, d, e, g, \alpha, \alpha_1, \eta, \delta, \delta_1, \beta_1, \beta_2,$ *and* ϵ *are real parameters. Then system (1) becomes* $(2\pi/\sqrt{7})$*-periodic. In this example, we show that if a matrix solution* $X_1(t)$ *of (9) is symplectic such that* rank$(X_1(t) - X(t)) = k \leq 3$ and $(X_1(t) - X(t))^T J (X_1(t) - X(t)) = 0$, *then we can find a matrix* $R \in \mathbb{R}^{6\times k}$ *of rank* $k \leq 3$, *whose columns generate an isotropic subspace and a diagonal matrix* $\Sigma = diag(s_1, s_2, s_3)$, *with* $sj = \pm 1$ *for all* $\boldsymbol{j} = 1, 2, 3$; *such that* $E(t)$ *be of the form (10), ie the following norm function*

$$\psi(t) = ||E(t) - ((J\, UR\Sigma R^T H(t))^T + J\, R\Sigma R^T H(t) + (R\Sigma R^T J)^T H(t)(R\Sigma R^T J))|| \text{ is close to zero}$$

for all $t \in [0, 2\pi/\sqrt{7}]$.

1) *For* $\epsilon = 2$, $\delta = 4$, $a = \beta_1 = 0.6272$, $b = 1.2544$, $d = 1.2544$, $d_1 = 1.4655$ *and* $e = g = \delta_1 = \alpha = \alpha_1 = \eta = \beta_1 = \beta_2 = 0$, *let's take*

$$X_1(0) = \begin{bmatrix} 1 & 0 & 0 & -0.6053 & 0 & 0 \\ 0 & 1 & 0 & 0 & -0.0002 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For all $t \in [0, 2\pi/\sqrt{7}]$, *we note in Figure 1 that*

$||X_1{}^T(t)JX_1(t) - J|| \leq 1.8 \times 10^{-14}$, $rg(X_1(t) - X(t)) = 2$, $||(X_1(t) - X(t))^T J(X_1(t) - X(t))|| = 2.5 \times 10^{-14}$, *where* $X(t)$ *is the solution of (2) and* $\psi(t) \leq 3 \times 10^{-16}$, *with*

$$R = \begin{bmatrix} 0.7780 & 0 \\ 0 & 0.0123 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad and \ \Sigma = I_2 \ such \ that \ R^T J R = 0.$$

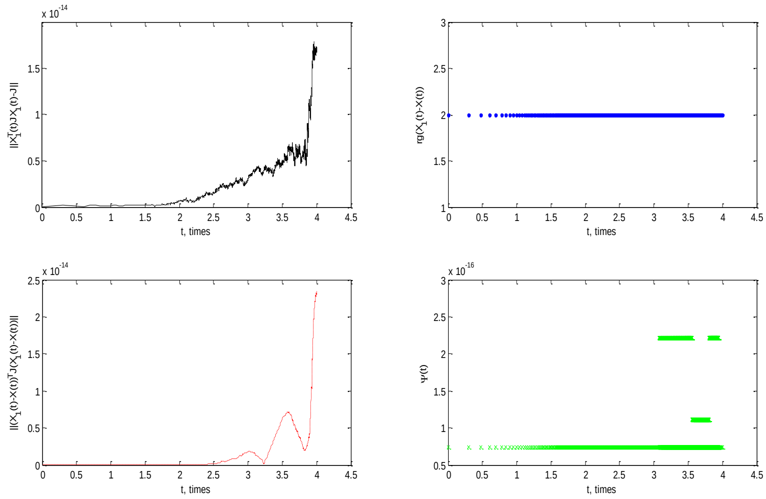

Figure 1 Checking of the proposition 3 for t∈[0, 2π/√7].

This shows that there exists a constant matrix $R \in \mathbb{R}^{6\times 2}$ *of rank* $k = 2$, *whose columns generate an isotropic subspace and a diagonal matrix* $\Sigma = \mathrm{diag}(1,1)$ *such that*

$$E(t) \equiv (J R\Sigma R^T H(t))^T + J R\Sigma R^T H(t) + (R\Sigma R^T J)^T H(t)(R\Sigma R^T J), \ \forall\, t \in [0, \tfrac{2\pi}{\sqrt{7}}].$$

2) For $\epsilon = 2$, $\delta = 4$, $a = 1.058 \times 10^{-3}$, $d = b = 2.116 \times 10^{-3}$, $e = 5.8368 \times 10^{-3}$, $g = 8.405 \times 10^{-2}$, $\alpha = 0.12608$, $\alpha_1 = 1.22646 \times 10^{-4}$, $\delta_1 = 0.0053$, $\eta = 2.9184 \times 10^{-3}$, $\beta_2 = 3.08768 \times 10^{-6}$, $\beta_1 = d_1 = 0$, *and*

$$X_1(0) = \begin{pmatrix} 1 & 0 & 0 & -0.0005 & 0 & 0 \\ 0 & 1 & 0 & 0 & -0.0420 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.0015 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we obtain Figure 2. In this Figure, we can see that $||X_1(t)JX_1(t) - J|| \leq 1.8 \times 10^{-14}$, $rg(X_1(t) - X(t)) = 3$,

$||(X_1(t) - X(t))^T J(X_1(t) - X(t))|| = 3.5 \times 10^{-16}$ *and* $\psi(t) \leq 1.15 \times 10^{-16}$, *with*

$$R = \begin{pmatrix} 0.023 & 0 & 0 \\ 0 & 0.205 & 0 \\ 0 & 0 & 0.0382 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ and \ \Sigma = I_3 \ such \ that \ R^T JR = 0 \ for \ all \ t \in [0, 2\pi/\sqrt{7}].$$

This proves that there exists a constant matrix $R \in \mathbb{R}^{6\times 3}$ of rank $k = 3$, whose columns generate an isotropic subspace and a diagonal matrix $\Sigma = diag(1,1,1)$ so that

$$E(t) \equiv (J R\Sigma R^T H(t))^T + J R\Sigma R^T H(t) + (R\Sigma R^T J)^T H(t)(R\Sigma R^T J), \ \forall\, t \in [0, \tfrac{2\pi}{\sqrt{7}}]$$

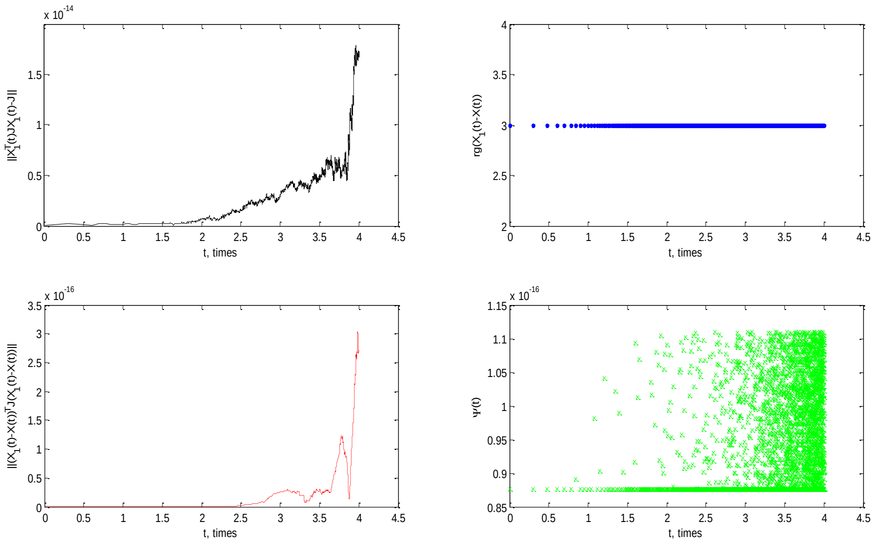

Figure 2 Checking of the proposition 3 for t ∈[0, 2π/√7].

CONCLUSION

In this research work, we first gave a generalization of lemma 7.1 of Mehl et al [9]. However, this result shows that any rank-k ($k \leq N$) perturbation of symplectic matrices can be in the form (3). Starting from this concept, we have also shown that if a solution $\tilde{X}(t)$ of (9), perturbation of (1) is symplectic such that $rank(\tilde{X}(t)-X(t))=k$ and $(\tilde{X}(t)-X(t))^T J\left(\tilde{X}(t)-X(t)\right)=0$ $for\ all\ t \in [0,\ P]$, then the matrix function $E(t)$ defined in (9) can be written in the form (10). Finally, two numerical examples allowed us to check our theoretical results. However, these examples show that these theoretical results are valid only for good choice of the perturbation. In future work, we will applied our results to the theory of parametric resonance in mechanic and physics.

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